MATH 211 (Section 2) - Midterm Project
Due October 19, 2004

Instructions.
This project is due on Wednesday, October 19. This project will be unpledged. It is an open-book, open-notes project. Feel free to discuss it with your colleagues, to consult with the instructor or the teaching assistants. However, the write up is your job, and you should do that much without any help.

Your grade on the project will reflect your mathematical skills as well as your communication skills. You will be expected to prepare your project submission with the same care as you do a paper in any other course. The only concession we make is that it is not necessary to type your paper, because typing mathematics and making it attractive and readable is quite difficult.

In preparing your paper, you should try to use good mathematical and scientific style. You should model your presentation on what you see in your textbooks in mathematics and in application areas. Your write-up should be designed to communicate your results to the reader. Approximately 20% of your grade will be based on the effectiveness with which you have done this. This means that you should:

- Include a separate cover sheet as the first page of your project containing the project title, your name, your instructor’s name, and your section number.
- Organize your write-up into paragraphs and complete sentences.
- Number your figures and refer to them by number in your text so the reader can find them when needed.
- Number the formulas that you develop and then refer to.
- Number your pages.
- Give proper credit to your sources by referencing them in footnotes or in a bibliography.
- Be sure to give reasons for your statements. Your arguments should be like those in the book. Make your explanations as concise as possible.
- Allow yourself enough time to transform your figures, formulas, and findings into a readable, understandable write-up.
Project description. We present (after S. Strogatz’s book ”Nonlinear dynamics and chaos”) a refinement of the model of a fishery with constant harvesting. If \( P(t) \) denotes the fish population at time \( t \), then the model to be considered is

\[
P' = rP(1 - P/K) - H \frac{P}{A + P}
\]

where \( r, K, A, P \) are positive constants. This model is more realistic in two respects: it has a fixed point at \( P = 0 \) for all values of the parameters, and the rate at which fish are caught decreases with \( P \). This is plausible–when fewer fish are available, it is harder to find them so the daily catch drops.

(a) Give a biological interpretation of the parameter \( A \); what does it measure?

(b) We try to reduce the number of parameters, by using the following dimensionless variables:

\[
x = \frac{P}{K}, \quad \tau = rt, \quad a = \frac{A}{K}, \quad h = \frac{H}{rK}.
\]

By applying the chain rule \( dx/d\tau = dx/dt \cdot dt/d\tau \), show that the system can be rewritten in dimensionless form as

\[
x' = x(1 - x) - h \frac{x}{a + x}.
\]

In this way, the new equivalent system has only two parameters. From now on, we are going to work with the dimensionless system.

(c) Assume \( h = 1/4 \). Analyze the number of fixed points and their stability properties as parameter \( a \) varies. Explain the long-term behavior of the population in each case. Plot in the \( ax \) plane the fixed points vs. \( a \). Can you pinpoint some values of \( a \) for which important changes in the behavior of the population happen? Such values of \( a \) are called bifurcation points.

(d) Assume \( a = 1/2 \). Analyze the number of fixed points and their stability properties as parameter \( h \) is varied. Explain the long-term behavior of the population in each case. Plot in the \( hx \) plane the fixed points vs. \( h \). Can you pinpoint some values of \( h \) for which important changes in the behavior of the population happen?

(e) Show that the system can have one, two, or three fixed points, depending on the values of \( a \) and \( h \). Classify the stability of the fixed points in each case. Explain the long-term behavior of the population in each case. Find in the parameter plane \( ah \) the curves that separate these various situations. Describe what happens on each of these curves and within the regions bounded by them.

(f) Assume now that \( h(t) = 1 + 3\sin(4t) \), and let \( a = 2 \). Use \textit{dfield} to study the long-term behavior of the system for different initial population values. Interpret your findings. Also, plot graphs comparing the approximation of the solution with initial condition \( x(0) = 2 \) on the interval \([0,6]\) using the \textit{ode45} procedure versus Euler’s method, RK2, and RK4 with step-size 0.5.