RESEARCH STATEMENT

1. INTRODUCTION

Overview. My research mainly focuses on the geometry and topology of two fundamental objects, namely the moduli space of surfaces and open non-positively curved manifolds of finite volumes. The projects in this statement involve a mixture of ideas from complex analysis, differential geometry, dynamical system, harmonic analysis, geometric topology, and geometric group theory. We mainly have the following three parts.

(a) Given a two dimensional surface $S$, there is a classical metric on the moduli space of $S$, namely the Weil-Petersson metric, which has been broadly studied over the past several decades by many mathematicians such as J. Brock, H. Masur, Y. Minsky, M. Mirzakhani, C. McMullen, S. Wolpert and M. Wolf. However, some fundamental questions are still very challenge in this field. I have recently been managing to study the Weil-Petersson metric from various aspects, for examples the asymptotics of Weil-Petersson geodesics, the curvature properties for large genus and the volume of balls with fixed radii for large genus. I will explain in section 2 how the results I obtained should be applicable to other fields like harmonic analysis, representation theory and CAT(0) geometry.

(b) The existence of a metric with nice curvature properties on the moduli space of surfaces is very interesting. For examples, the Brock-Farb-McMullen question asks whether there exists a complete non-positively curved Riemannian metric on the moduli space of surfaces; the Farb-Weinberger conjecture asserts that there does not exist a complete Riemannian metric of positive scalar curvature on the moduli space of surfaces such that the metric is quasi-isometric to the classical Teichmüller metric. In section 4, I will describe the status of these questions.

(c) Gromov’s celebrated Tameness Theorem says that a complete open negatively curved manifold with bounded curvature of finite volume has finite type, i.e., it is homotopy equivalent to a compact manifold with boundary. A number of conjectures on this geometric object have been posted since 1980’s. Some of them can be addressed by my recent work. In section 3, I will describe a number of projects about different aspects of this geometric object.
2. TEICHMÜLLER SPACE AND THE WEIL-PETERSSON METRIC

2.1. Nonpositivity of the Riemannian curvature operator of the Weil-Petersson metric and its application. Let $S = S_g$ be a closed surface of genus $g > 1$ and $\text{Teich}(S)$ be the Teichmüller space of $S_g$ endowed with the Weil-Petersson metric. One celebrated result is the Weil-Petersson curvature formula which was established by Tromba [Tro86] and Wolpert [Wol86]. The curvature of $\text{Teich}(S)$ has been studied over the past several decades by using the Tromba-Wolpert curvature formula. For example; the space $\text{Teich}(S)$ has negative sectional curvature [Tro86, Wol86], strongly negative curvature in the sense of Siu [Sch86], and dual Nakano negative curvature [LSY08].

Let $X \in \text{Teich}(S)$ and $\wedge^2 T_X(\text{Teich}(S))$ be the exterior wedge of its tangent spaces at $X$, and let $Q$ be the Riemannian sectional curvature operator of $\text{Teich}(S)$. From the viewpoint of matrix theory, Tromba-Wolpert’s theorem says that the diagonal entries of the matrix are negative. Recently, I established the following result which asserts that the matrix is non-positive definite. More precisely,

**Theorem 2.1 (Wu, [Wu14]).** $\text{Teich}(S)$ has non-positive sectional curvature operator. Moreover, $Q(A, A) = 0$ if and only if there exists an element $B$ in $\wedge^2 T_X\text{Teich}(S)$ such that $A = B - J \circ B$ where $J$ is the almost complex structure on $\text{Teich}(S)$ and $J \circ B$ is the natural action of the almost complex structure on the exterior wedge of the tangent spaces.

A harmonic map between two spaces is a critical point of the energy functional. When the domain is the Quaternionic hyperbolic space or the Cayley plane, various rigidity theorems were established in [Cor92, GS92, JY97, MSY93]. When the target is the Teichmüller space endowed with the Weil-Petersson metric, we proved the following rigidity theorem.

**Theorem 2.2 (Wu, [Wu14]).** Let $\Gamma$ be a lattice in a semisimple Lie group $G$ which is either $\text{Sp}(m, 1)$ or $E^{-20}_4$, and $\text{Mod}(S_g)$ be the mapping class group of $\text{Teich}(S)$. Then, any twisted harmonic map $f$ from $G/\Gamma$ into $\text{Teich}(S)$ with respect to a homomorphism $\rho : \Gamma \to \text{Mod}(S_g)$ must be a constant.

Theorem 2.2 is applied by Daskalopoulos and Mese in [DM15] to give an analytic proof for Ivanov-Farb-Kaimanovich-Mazur-Yeung’s theorem which says that a cocompact lattice of any symmetric space with non-positive curvature other than the real or complex hyperbolic spaces $\Gamma$ can only have finite image for any homomorphism $\Gamma \to \text{Mod}(S_g)$. Let $\overline{\text{Teich}(S)}$ be the completion of $\text{Teich}(S)$ by adding different strata consisting of nodal curves. The following question is raised in these projects.

**Question 2.3.** Let $M^n$ ($n \geq 3$) be a Riemannian manifold and $f : M \to \overline{\text{Teich}(S)}$ be a harmonic map. Is the image $f(M)$ just contained in a single stratum of $\overline{\text{Teich}(S)}$?
2.2. **Uniform bounds for Weil-Petersson curvatures.** The Weil-Petersson curvature on the thick part of the Teichmüller space is studied by Huang [Hua07] and Teo [Teo09]. The question, whether the thick parts of the moduli spaces of surfaces are becoming increasingly flat in some sense of the Weil-Petersson geometry as the genus tends to infinity which is asked by M. Mirzakhani, is open for years. Recently, jointly with Michael Wolf, we answered this question from the viewpoints of holomorphic sectional curvature and Riemannian sectional curvature operator. More precisely,

**Theorem 2.4** (Wolf-W, [WW15]). Let \( \{X_g\} \subset \text{Teich}(S_g) \) be a sequence of hyperbolic surfaces with sufficiently large injectivity radii. Then, the Weil-Petersson holomorphic sectional curvature \( \text{HolK} \) satisfies

\[
\min_{\sigma_g \subset T_{X_g}\text{Teich}(S_g)} \text{HolK}(\sigma_g) \asymp -1.
\]

Where \( f_1(g) \asymp f_2(g) \) means that there exists a universal constant \( C > 0 \), independent of \( g \), such that \( f_2(g) \geq C f_1(g) \).

**Theorem 2.5** (Wolf-W, [WW15]). Given any positive \( \epsilon_0 > 0 \). Then for any \( X_g \in \text{Teich}(S_g)^{\geq \epsilon_0} \) (the \( \epsilon_0 \)-thick part of the Teichmüller space), the \( \ell^\infty \)-norm of the Riemannian Weil-Petersson curvature operator \( Q \) at \( X_g \) satisfies that

\[
||Q||_{\ell^\infty}(X_g) \asymp 1.
\]

Theorem 2.1 tells that there exist exactly \( (3g - 3)^2 \) nonzero eigenvalues for the curvature operator \( Q \) at each \( X \in \text{Teich}(S_g) \). We define the \( \ell^p \)-norm \( ||Q||_{\ell^p}(X) \) of the Weil-Petersson curvature operator at \( X \) as

\[
||Q||_{\ell^p}(X) := \left( \sum_{i=1}^{(3g-3)^2} |\lambda_i(X)|^p \right)^{\frac{1}{p}} \quad (1 \leq p \leq \infty).
\]

The following question seems to require us to understand the tangent space of \( \text{Teich}(S_g) \) at the thick part for large genus at a deeper level.

**Question 2.6.** What can we say about \( ||Q||_{\ell^p}(X_g) \) as \( g \) goes to infinity when \( \{X_g\} \subset \text{Teich}(S_g)^{\geq \epsilon_0} \)?

When \( p = 1 \), by the results in Wolpert [Wol86], Teo [Teo09] and Theorem 2.1 we know that \( ||Q||_{\ell^1}(X_g) \asymp g \). When \( p = \infty \), Theorem 2.5 gives that \( ||Q||_{\ell^\infty}(X_g) \asymp 1 \). An upper bound for \( ||Q||_{\ell^p}(X_g) \) is given in [WW15]. We do not know whether this upper bound is the right answer to Question 2.6.

2.3. **Weil-Petersson curvatures for large genus.** The subject of the asymptotic geometry of \( \mathcal{M}_g \) as \( g \) tends to infinity, has recently become quite active: see for example Mirzakhani [Mir07a, Mir07b, Mir13] for the volume of \( \mathcal{M}_g \), Cavendish-Parlier [CP12] for the diameter of \( \mathcal{M}_g \) and Bromberg-Brock [BB] for the least Weil-Petersson translation length of pseudo-Anosov mapping classes. Recently we showed that the rate \( -\frac{1}{g} \), lying in Tromba-Wolpert’s upper bound [Wol86, Tro86] for Weil-Petersson holomorphic sectional curvature, is optimal as \( g \) tends to infinity. More precisely,
Theorem 2.7 (Wu, [Wu15a]). Given a constant $\epsilon_0 > 2\ln(3 + 2\sqrt{2})$ and let $\{X_g\} \subset \text{Teich}(S_g)^{>0}$. Then, the Weil-Petersson holomorphic sectional curvature $\text{HolK}$ at $X_g$ satisfies
\[ \max_{\nu \in T_{X_g}\text{Teich}(S_g)} \text{HolK}(\nu) \asymp -\frac{1}{g}. \]

Let $\mathcal{M}_g$ be the moduli space of surfaces. For any two dimensional plane $P \subset T_{X_g}\mathcal{M}_g$ (maybe not holomorphic), we denote by $K(P)$ the Riemannian Weil-Petersson sectional curvature of the plane $P$. The following result tells that, in a probabilistic way, the minimal Riemannian sectional curvature is uniformly negative on $\mathcal{M}_g$ as $g$ tends to infinity. More precisely,

Theorem 2.8 (Wu, [Wu15a]). There exists a universal constant $C_0 > 0$ such that the probability satisfies
\[ \lim_{g \to \infty} \text{Prob}\{X_g \in \mathcal{M}_g; \min_{P \subset T_{X_g}\mathcal{M}_g} K(P) \leq -C_0 < 0\} = 1. \]

Since $\mathcal{M}_g$ has negative sectional curvature [Wol86, Tro86], the following function $h$ is well-defined.
\[ h(X_g) := \min_{P \subset T_{X_g}\mathcal{M}_g} K(P) \max_{P \subset T_{X_g}\mathcal{M}_g} K(P), \quad \forall X_g \in \mathcal{M}_g. \]

It is clear that $h(X_g) \geq 1$ for all $X_g \in \mathcal{M}_g$. The results in [Hua05, Wol08] tell that $\sup_{X_g \in \mathcal{M}_g} h(X_g) = \infty$, which happens near a particular boundary of $\mathcal{M}_g$. However, it is not clear about the range of $h$ in the thick part of the moduli space. Our next result is that, in a probabilistic way, the function $h$ is unbounded globally on $\mathcal{M}_g$ as $g$ tends to infinity. More precisely,

Theorem 2.9 (Wu, [Wu15a]). For any $L > 0$, then the probability satisfies
\[ \lim_{g \to \infty} \text{Prob}\{X_g \in \mathcal{M}_g; h(X_g) \geq L\} = 1. \]

The proofs of Theorem 2.8 and 2.9 rely on some recent probabilistic results of M. Mirzakhani in [Mir13]. Indeed, the following question is very interesting.

Question 2.10. Can we drop the probability in Theorem 2.8 and 2.9? For example in Theorem 2.8, can we find a universal negative upper bound for the minimal Weil-Petersson sectional curvature at every point of $\mathcal{M}_g$?

2.4. Limits of Weil-Petersson geodesics. Let $T(S)$ be the Teichmüller space of $S$ (without metric), and let $X, Y \in T(S)$ and $Q(X, Y)$ be the quasi-Fuchsian hyperbolic 3-manifold determined by $X$ and $Y$. In [Bro01] Brock shows that for any $\phi \in \text{Mod}(S_g)$, there is an $s \geq 1$ depending only on $\phi$ and bounded in terms of $S$ so that the sequence $\{Q(\phi^{n_s}\circ X, Y)\}_{n \geq 1}$ converges algebraically and geometrically. The following recent result is analogous to Brock’s Theorem in the setting of Weil-Petersson geometry.

Theorem 2.11 (Wu, [Wu12]). Let $\phi \in \text{Mod}(S_g)$ be a mapping class. Then there is an $s \geq 1$ only depending on $\phi$ so that the sequence of the directions of the geodesics $\{g(X, \phi^{n_s}\circ Y)\}_{n \geq 1}$ is convergent in the visual sphere of $X$. 
For $\phi \in \text{Mod}(S)$, we define $|\phi| := \inf_{X \in \text{Teich}(S)} \text{dist}(X, \phi \circ X)$.

If $\phi$ is reducible and $|\phi| = 0$, then up to some power, $\phi$ is a multi Dehn-twist. The limit geodesic in Theorem 2.11, after being projected onto the moduli space $\mathbb{M}_g$, is a piecewise geodesic whose vertices (except the two endpoints) lie in the strata whose vanishing curves belong to the ones on which $\phi$ is a multi Dehn-twist. More precisely,

**Theorem 2.12** (Wu, [Wu12]). Let $\sigma$ be a $m$-simplex and $\sigma^0 = \{\alpha_1, \cdots, \alpha_{m+1}\}$ and $\tau_i$ be the Dehn-twist about the curve $\alpha_i$ for $i = 1, 2, \cdots, m+1$. Let $\phi = \prod_{1 \leq i \leq m+1} \tau_i \in \text{Mod}(S_g)$ and $X, Y \in \text{Teich}(S)$, and $g_n$ be the unit speed geodesic $g(X, \phi^n \circ Y)$. Then, there exist a positive number $L$; an associated partition $0 = t_0 < t_1 < \cdots < t_k = L$; simplices $\sigma_0, \cdots, \sigma_k$; and a piecewise geodesic

$$g : [0, L] \to \text{Teich}(S)$$

with the following properties.

(1). $\sigma^0_i \subset \sigma^0, \sigma^0_i \cap \sigma^0_j$ is empty for $i \neq j$,

(2). $\sigma^0 = \bigcup_{i=1}^{k} \sigma^0_i$,

(3). $g(t_i) \in T_{\sigma_i}, i = 1, \cdots, k-1, g(0) = X, g(t_k) = Y$,

(4). $g_n[0, t_1]$ converges in $\text{Teich}(S)$ to the restriction $g([0, t_1])$, and for each $i = 1, \cdots, k-1$,

$$\lim_{n \to +\infty} \text{dist}(\tau_{i,n} \circ \cdots \circ \tau_{1,n} \circ g_n(t), g(t)) = 0, \text{ for } t \in [t_i, t_{i+1}].$$

Where $\tau_{i,n} = \prod_{\alpha \in \sigma_i} \tau_{\alpha,n}^{-n}$, for $i = 1, \cdots, k-1$.

(5). The piecewise geodesic $g$ is the unique minimal length path in $\text{Teich}(S)$ joining $g(0)$ to $g(L)$ and intersecting the closures of the strata $T_{\sigma_1}, T_{\sigma_2}, \cdots, T_{\sigma_{k-1}}$ in order.

If $m = 0$, Theorem 2.12 was proven by Brock in [Bro05] (one can also see [Wol10]). The convexity of the length function of closed curve along the Weil-Petersson geodesic [Wol87, Wol12] is crucial to the proof.

If $\phi$ is irreducible and $|\phi| > 0$, by using the method in [BMM11] we can show that the limit ray in Theorem 2.11 is strongly asymptotic to the axis of $\phi$, which stays in the thick part of $\text{Teich}(S)$.

If $\phi$ is reducible and $|\phi| > 0$, the limit geodesic of $\{g(X, \phi^n \circ Y)\}$ goes to an explicit stratum which only depends on $\phi$. More precisely,

**Theorem 2.13** (Wu, [Wu12]). Let $\phi \in \text{Mod}(S_g)$ be reducible with $|\phi| > 0$ and $k$ be a positive integer such that $\phi^k = \prod_{\alpha \in \sigma} \tau_{\alpha} \times \prod_{j} \phi_j$ where $\sigma$ is a simplex, $\tau_{\alpha}$ is Dehn-twist about $\alpha$ and $\phi_j = \phi^{k}|_{PS_j}$ is pseudo-Anosov on $PS_j$. Here $PS_j$ is proper subsurface of $S$. Then for any $X, Y \in \text{Teich}(S)$, there exists a geodesic ray $c : [0, +\infty) \to \text{Teich}(S)$ such that

(1). The sequence $\{g(X, \phi^n \circ Y)\}$ converges to $c : [0, +\infty) \to \text{Teich}(S)$.
(2). For any simple closed curve $\alpha \in \partial(\bigcup j PS_j)$, we have

$$\lim_{t \to +\infty} \ell_\alpha(c(t)) = 0.$$ 

(3). There exists a positive number $\epsilon_0$ such that for any non peripheral essential simple closed curve $\beta$ in $S$ but not in $\partial(\bigcup j PS_j)$,

$$\ell_\beta(c(t)) \geq \epsilon_0$$

for all $t \geq 0$.

An unexpected outcome of Theorem 2.13 is that the twist part of $\phi$ does not affect the limit ray. A natural question from the theorem above is

**Problem 2.14.** Under what conditions can Theorem 2.13 be extended to allow $X$ to be also iterated by mapping classes?

Consider the sequence of geodesics $\{g(\psi^n \circ X, \phi^n \circ Y)\}_{n \geq 1}$. If $\psi = \phi$, this sequence will leave every compact subset. If $\psi$ is pseudo-anosov, the methods in [BMM11] will guarantee the existence of the limit. For other cases, the most difficult point for problem 2.14 may be to study the following conjecture.

**Conjecture 2.15.** Let $\alpha_1$ and $\alpha_2$ be two simple closed curves which fill the surface $S$. The geodesic rays $c_i (i = 1, 2)$ satisfy that $\ell_{\alpha_i}(c_i(t)) \to 0$ as $t \to \infty$. Then there exists a geodesic line in $\text{Teich}(S)$ joining $c_1$ and $c_2$.

The conjecture above may involve studying the rate of growth of the length function of a simple closed curve along geodesic rays.

### 2.5. Weil-Petersson volume of geodesic ball of fixed radius for large genus.

The asymptotic behavior of the volume of $M_g$ as $g$ tends to infinity, has been quite well studied in [Pen92, Mir07a, Mir07b, Mir13, ST01, Zog08]. The following question is raised by M. Mirzakhani.

**Question 2.16.** Fix a constant $R > 0$. Let $X_g \in \text{Teich}(S_g)$ and $B(X_g; R) \subset \text{Teich}(S_g)$ be the geodesic ball centered at $X_g$ of radius $R$. What can we say about the volume $\text{Vol}_{WP}(B(X_g; R))$ as $g$ goes to infinity?

Since $\text{Teich}(S_g)$ is incomplete [Chu76, Wol75], it is not hard to see that for each $g \geq 2$ we have $\sup_{X_g \in \text{Teich}(S_g)} \text{Vol}_{WP}(B(X_g; R)) = +\infty$. The case of the infimum seems to be interesting. The following result may give an upper bound for the infimum. More precisely,

**Theorem 2.17** (Wu, [Wu]). For any fixed constant $R > 0$ and any constant $\epsilon > 0$ we have

$$\inf_{X_g \in \text{Teich}(S_g)} \text{Vol}_{WP}(B(X_g; R)) = o\left(\frac{1}{g}\right)^{(3-\epsilon)g}.$$ 

In particular, $\inf_{X_g \in \text{Teich}(S_g)} \text{Vol}_{WP}(B(X_g; R)) \to 0$ as $g \to \infty$. 
2.6. The curvature operator on the universal Teichmüller space.

In [TT06] Teo and Takhtajan introduced the Weil-Petersson metric on the universal Teichmüller space $T(1)$. They also established a curvature formula which is similar to Tromba-Wolpert’s formula for the Teichmüller space of finite-type surfaces. For any $X \in T(1)$, the Weil-Petersson Riemannian curvature operator $\tilde{Q}$ at $X$ can be viewed as a symmetric linear self-operator of a Hilbert space of infinite dimension. Motivated by the methods in [Wu14, WW15], jointly with Zheng Huang we have the following result.

**Theorem 2.18** (Huang-W, [HW]). Let $T(1)$ be the universal Teichmüller space and $X \in T(1)$. Then, the Weil-Petersson Riemannian curvature operator $\tilde{Q}$ of the universal Teichmüller space at $X$ satisfies

1. $\tilde{Q}$ is non-positive definite.
2. $\tilde{Q}$ is a bounded operator.
3. $\tilde{Q}$ is not compact: the set of the negative spectra of $\tilde{Q}$ is not discrete.

It is interesting to know whether Theorem 2.2 still holds if one replaces $\text{Teich}(S)$ by $T(1)$. We believe Theorem 2.18 is helpful for this question.

3. Obstructions on metrics

3.1. Riemannian metric of positive scalar curvature. One result of Gromov-Lawson in [GL83] states that given a complete Riemannian manifold $(X, ds^2_1)$ of non-positive sectional curvature, then $X$ cannot admit any Riemannian metric $ds^2_2$ on $X$ with $ds^2_2 > ds^2_1$ such that $(X, ds^2_2)$ has positive scalar curvature where $ds^2_2 > ds^2_1$ means that $ds^2_2 \geq k \cdot ds^2_1$ for some constant $k > 0$. In [Far06] Farb-Weinberger shows that any finite cover of the moduli space of surfaces admits a complete finite-volume Riemannian metric of uniformly positive scalar curvature. It is known that the Teichmüller metric $ds^2_T$ is not non-positively curved [Mas76]. Recently jointly with Kefeng Liu, we use some recent accomplishments in [McM00, LSY04, LSY05, BBF14] on the geometry of Teichmüller space as bridges to prove the following result which is analogous to Gromov-Lawson’s theorem in the Teichmüller setting.

**Theorem 3.1** (Liu-W, [LW15]). Let $S_g$ be a closed Riemann surface of genus $g$ with $g \geq 2$ and $M$ be a finite cover of the moduli space $\mathcal{M}_g$ of $S_g$. Then for any Riemannian metric $ds^2$ on $M$ with $ds^2 > ds^2_T$ we have

$$\inf_{p \in (M, ds^2)} \text{Sca}(p) < 0.$$  

It was conjectured in [Far06] (see Conjecture 4.6 in [Far06]) that any finite cover $M$ of the moduli space $\mathcal{M}_g$ of $S_g$ does not admit a finite-volume Riemannian metric of uniformly positive scalar curvature in the quasi-isometry class of the Teichmüller metric. The second result in [LW15] provides a proof for this conjecture.

**Theorem 3.2** (Liu-W, [LW15]). Let $S_g$ be a closed surface of genus $g$ with $g \geq 2$. Then any cover $M$ of the moduli space $\mathcal{M}_g$ of $S_g$ does not admit a
complete Riemannian metric of uniformly positive scalar curvature in the quasi-isometry class of the Teichmüller metric.

There are no conditions on finite cover or finite volume in Theorem 3.7, compared to the Farb-Weinberger conjecture. Moreover, the following interesting question was raised during this project.

**Question 3.3.** Is \((T(S_g), ds^2_T)\) hyperspherical in the sense of Gromov-Lawson in [GL83]? Or, for any \(\epsilon > 0\) does there exist an \(\epsilon\)-contraction diffeomorphism from \((T(S_g), ds^2_T)\) to the Euclidean space \(\mathbb{R}^{6g-6}\)?

A positive answer to the question above would give a new parametrization for the Teichmüller space.

### 3.2. Hermitian metric of non-negative scalar curvature.

As stated above, Farb and Weinberger proved the existence of complete finite-volume Riemannian metric of uniformly positive scalar curvature on the moduli space. The following result asserts that this can not happen for Hermitian metrics.

**Theorem 3.4** (Wu, [Wu15b]). A finite cover of the moduli space of surfaces does not admit any finite-volume Hermitian metric of nonnegative scalar curvature.

The Asymptotic Poincaré metric, Induced Bergman metric, Kähler-Einstein metric, McMullen metric, Ricci metric, and perturbed Ricci metric are complete and Kähler. In [LSY04, LSY05, McM00, Yeu05], the authors showed that the metrics listed above are equivalent to the Teichmüller metric. Theorem 3.4 implies that the scalar curvatures of these metrics have to be negative at some point. Indeed, the following theorem asserts that the total scalar curvature is negative for any one of these nice metrics.

**Theorem 3.5** (Wu, [Wu15b]). Let \(M \) be a finite cover of the moduli space \(\mathcal{M}_g\) of \(S_g\) and \(ds^2\) be an almost Hermitian metric on \(M\) with \(ds^2 \asymp ds^2_T\). Assuming that the scalar curvature of \((M, ds^2)\) is bounded from below, then we have

\[
\int_{p \in (M, ds^2)} \text{Sca}(p) \, d\text{Vol}(p) < 0.
\]

### 3.3. Riemannian metric of non-positive sectional curvature.

Let \(\mathcal{M}(S_{g,n})\) be the moduli space of surface with \(g\) genus of \(n\) punctures. There are many canonical metrics on \(\mathcal{M}(S_{g,n})\) such as the Teichmüller metric, the Weil-Petersson metric, and so on. As stated above, Masur in [Mas75] proved that the Teichmüller metric is not non-positively curved except in a few several cases. The Weil-Petersson metric is negatively curved, but not complete. There is a question in Brock-Farb’s paper [BF06] which asks

**Question 3.6** (Brock-Farb-McMullen). Does \(\mathcal{M}(S_{g,n})\) admit a complete, non-positively curved Riemannian metric?
It is not hard to see that the question above is reduced to the cases $3g + n \geq 5$. We say a non-positively curved space $M$ is visible if for any two different points in the visual boundary of the universal covering of $M$, those points can be joined by a geodesic line.

**Theorem 3.7** (Wu, [Wu15c]). If $3g + n \geq 5$, then $M(S_{g,n})$ admits no complete visible $\text{CAT}(0)$ Riemannian metric.

Question 3.6 is wide open, even for the cases such as the space having finite volume and the curvature lies in $[-1, 0)$, or within the equivalent class of the Teichmüller metric. Exploring problem 3.6 seems to require a next level for subtlety in understanding both Teichmüller theory and theory on the fundamental groups of complete open non-positively curved manifolds, which is one of my long-term goals.

4. Geometry and topology of open non-positively curved metric spaces

4.1. Parabolic isometries on a visible $\text{CAT}(0)$ space. $\text{CAT}(0)$ spaces are generalizations of Riemannian manifolds with non-positive sectional curvature to geodesic spaces. Let $M$ be a $\text{CAT}(0)$ space. An isometry $\gamma$ of $M$ is called parabolic provided that the translation length $|\gamma| := \inf_{x \in M} \text{dist}(\gamma \circ x, x)$ cannot be achieved in $M$. The following result may characterize parabolic isometry of positive translation length on visible $\text{CAT}(0)$ space.

**Theorem 4.1** (Wu, [Wu11]). Let $M$ be a complete visible $\text{CAT}(0)$ space. Then $\gamma$ is a parabolic isometry of $M$ with translation length $|\gamma| > 0$ if and only if there exists an infinite-flat-strip $U \times \mathbb{R}$, which is a closed convex subset of $M$, such that $\gamma$ acts on $U \times \mathbb{R}$ as

$$\gamma \cdot (x, t) = (\gamma_1 \cdot x, t + t_0),$$

where $\gamma_1$ is a parabolic isometry of $U$ with $|\gamma_1| = 0$ and $t_0 \neq 0$.

It is not hard to see that both proper $\text{CAT}(0)$ spaces and Gromov-Hyperbolic $\text{CAT}(0)$ spaces cannot have any infinite-flat-strip. Theorem 4.1 implies any parabolic isometry of a proper $\text{CAT}(0)$ space or a Gromov-Hyperbolic $\text{CAT}(0)$ space has zero translation length. For the Gromov-Hyperbolic case, this result is due to Buyalo in [Buy98]. For the proper case, Theorem 4.1 implies a conjecture of Phan in [Tam11] asserting that a tame, finite-volume, negatively curved manifold $M$ is not visible if its fundamental group $\pi_1(M)$ contains a parabolic isometry of $\tilde{M}$ with positive translation length, which was posed around the same time when Theorem 4.1 was proved.

4.2. Geometry and topology of manifolds with sectional curvatures in $[-1, 0]$ and finite volumes. Gromov’s celebrated Tameness theorem in [Gro78] states that a finite-volume Riemannian manifold $M$ with sectional curvature in $[-1,0)$ has finite type, i.e., $M$ is homotopy equivalent to a compact manifold with boundary. During the same time, Eberlein in [Ebe80]
showed that a finite-volume visible Riemannian manifold \( M \) with sectional curvature in \([-1,0]\) has finite type. He also conjectured [Ebe80] that

**Conjecture 4.2** (Eberlein). A finite-volume Riemannian manifold \( M \) with sectional curvature in \([-1,0)\) is visible.

The answer to the conjecture is negative, which is well-known to experts because of Buyalo’s example in [Buy93] for certain topological reason. We use Theorem 4.1 to give a geometric reason to disprove Eberlein’s conjecture.

**Theorem 4.3** (Wu, [Wu11]). The fundamental group of the manifold constructed in [Fuj88] with finite volume and sectional curvature in \([-1,0)\) contains a parabolic isometry of the universal covers with positive translation length. In particular, they are not visible.

For a finitely generated group \( \Gamma \), Bestvina, Kapovich and Kleiner in [BKK02] introduce the action dimension of \( \Gamma \) which is the smallest dimension of contractible manifold on which \( \Gamma \) properly acts on. Meanwhile, Bestvina and Feighn in [BF02] proves that the action dimension of any lattice of a locally symmetric no-compact manifold is equal to the dimension of manifold.

**Question 4.4.** Let \( M \) be a finite-volume Riemannian manifold with curvature in \([-1,0)\). Is the action dimension of the fundamental group of \( M \) equal to the dimension of \( M \)?

For the negatively pinched case it is not hard to use the structure of any end of \( M \) to show that these two dimensions coincide. For low dimensions, by using the Gauss-Bonnet-Chern formula and \( L^2 \)-cohomology theory, jointly with Avramidi and Phan, we prove the following theorem.

**Theorem 4.5** (Avramidi-Phan-W, [ATW13]). Let \( M^n \) be a finite-volume, \( n \)-dimensional Riemannian manifold with curvature in \([-1,0)\). Then, if \( n \leq 4 \), the action dimension of the fundamental group of \( M^n \) is equal to \( n \).

Actually the following special case of Question 4.4 is not known yet.

**Question 4.6.** Let \( M^n \) (\( n \geq 4 \)) be a closed aspherical manifold. Does \( M^n \times \mathbb{R} \) admit a complete finite-volume Riemannian metric such that the sectional curvature is in \([-1,0)\)?

We believe the obstructor dimension, developed in [BKK02], is useful in studying problem 4.4 and 4.6.

**References**


[Mas75] Howard Masur, On a class of geodesics in Teichmüller space, Ann. of Math. (2) 102 (1975), no. 2, 205–221.

RESEARCH STATEMENT


