Section 1. Vectors and Matrices

1.1. The geometric sum

Analytically,

\[ x + y = \left( \begin{array}{c} 1 \\ -2 \end{array} \right) + \left( \begin{array}{c} -3 \\ 1 \end{array} \right) = \left( \begin{array}{c} -2 \\ 3 \end{array} \right) \]

1.2. The geometric sum

Analytically,

\[ x + y = \left( \begin{array}{c} 4 \\ -2 \end{array} \right) + \left( \begin{array}{c} -3 \\ -5 \end{array} \right) = \left( \begin{array}{c} 1 \\ -7 \end{array} \right) \]
1.3. The geometric solution

Analytically,

\[ 2x = 2 \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix} \]

1.4. The geometric solution

Analytically,

\[ -3x = -3 \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix} \]

1.5. The geometric solution

\[ 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \end{pmatrix} \]
Checking,

\[ 2x + 1y = 2 \left( \begin{array} {c} 1 \\ 2 \\ \end{array} \right) + 1 \left( \begin{array} {c} -3 \\ 4 \\ \end{array} \right) = \left( \begin{array} {c} -1 \\ 8 \\ \end{array} \right) \]

1.6. The geometric solution

\[ 2 \left( \begin{array} {c} 1 \\ 1 \\ \end{array} \right) + 3 \left( \begin{array} {c} 1 \\ -1 \\ \end{array} \right) = \left( \begin{array} {c} 5 \\ -1 \\ \end{array} \right) \]

Checking,

\[ 2x + 3y = 2 \left( \begin{array} {c} 1 \\ 1 \\ \end{array} \right) + 3 \left( \begin{array} {c} 1 \\ -1 \\ \end{array} \right) = \left( \begin{array} {c} 5 \\ -1 \\ \end{array} \right) \]

1.7. The geometric solution

\[ -2 \left( \begin{array} {c} 1 \\ 3 \\ \end{array} \right) + 2 \left( \begin{array} {c} 4 \\ 1 \\ \end{array} \right) = \left( \begin{array} {c} 6 \\ -4 \\ \end{array} \right) \]

Checking,

\[ -2x + 2y = -2 \left( \begin{array} {c} 1 \\ 3 \\ \end{array} \right) + 2 \left( \begin{array} {c} 4 \\ 1 \\ \end{array} \right) = 1 \left( \begin{array} {c} 6 \\ -4 \\ \end{array} \right) \]

1.8. You cannot write \( z \) as a linear combination of \( x \) and \( y \). It does not lie on the line spanned by \( x \) and \( y \).
The geometric solution

\[
2x + 1y = 2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

1.9. If you multiply matrix \( A \) by \([2, 0, 0]^T\), then

\[
A \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = [a_1, a_2, a_3] \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 2a_1.
\]

You can use similar strategies to triple and quadruple the second and third columns, respectively. Thus,

\[
\begin{pmatrix} -1 & 2 & 4 \\ 0 & 5 & 2 \\ -1 & -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 6 & 16 \\ 0 & 15 & 8 \\ -2 & -6 & 12 \end{pmatrix}.
\]

The columns of \( AB \) are

\[
Ab_1 = [a_1, a_2, a_3] \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 2a_1,
\]

\[
Ab_2 = [a_1, a_2, a_3] \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = 3a_2,
\]

\[
Ab_3 = [a_1, a_2, a_3] \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = 4a_3.
\]

Thus,

\[
B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.
\]

1.10. First,

\[
A(x + y) = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \left( \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix} \right),
\]

\[
= \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix},
\]

\[
= \begin{pmatrix} 3 \\ -3 \end{pmatrix}.
\]
But,

\[
Ax + Ay = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.
\]

Secondly,

\[
A(ax) = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix},
\]

\[
= \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -4 \end{pmatrix},
\]

\[
= \begin{pmatrix} -10 \\ 6 \end{pmatrix}.
\]

But

\[
a(Ax) = 2 \left( \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right)
\]

\[
= 2 \begin{pmatrix} -5 \\ 6 \end{pmatrix} = \begin{pmatrix} -10 \\ 6 \end{pmatrix}.
\]

1.11. First,

\[
A(ax) = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 15 \\ 6 \end{pmatrix}.
\]

But,

\[
a(Ax) = -3 \left( \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right)
\]

\[
= -3 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ 6 \end{pmatrix}.
\]

Secondly,

\[
A(x + y) = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix},
\]

\[
= \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix},
\]

\[
= \begin{pmatrix} -7 \\ 0 \end{pmatrix}.
\]

But,

\[
Ax + Ay = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix},
\]

\[
= \begin{pmatrix} -5 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \end{pmatrix}
\]

\[
= \begin{pmatrix} -7 \\ -1 \end{pmatrix}.
\]

1.12. (a) If \((x_1, \ldots, x_n)^T, (y_1, \ldots, y_n)^T \in \mathbb{R}^n\), then \((x_1 + y_1, \ldots, x_n + y_n)^T \in \mathbb{R}^n\).
(b) If \(a \in \mathbb{R}\) and \((x_1, \ldots, x_n)^T \in \mathbb{R}^n\), then \((ax_1, \ldots, ax_n)^T \in \mathbb{R}^n\).
(c) If \((x_1, \ldots, x_n)^T, (y_1, \ldots, y_n)^T \in \mathbb{R}^n\), then \((x_1 + y_1, \ldots, x_n + y_n)^T = (y_1 + x_1, \ldots, y_n + x_n)^T\).

(d) If \((x_1, \ldots, x_n)^T, (y_1, \ldots, y_n)^T, (z_1, \ldots, z_n)^T \in \mathbb{R}^n\), then \(((x_1 + y_1) + z_1, \ldots, (x_n + y_n) + z_n)^T = (x_1 + (y_1 + z_1), \ldots, x_n + (y_n + z_n))^T\).

(e) If \((x_1, \ldots, x_n)^T \in \mathbb{R}^n\), then \((0, 0, \ldots, 0)^T \in \mathbb{R}^n\) and \((x_1 + 0, \ldots, x_n + 0)^T = (x_1, \ldots, x_n)^T\).

(f) If \((x_1, \ldots, x_n)^T \in \mathbb{R}^n\), then \((-x_1, \ldots, -x_n)^T \in \mathbb{R}^n\) and \((x_1 + (-x_1), \ldots, x_n + (-x_n))^T = (0, \ldots, 0)^T\).

(g) If \(\alpha \in \mathbb{R}, (x_1, \ldots, x_n)^T, (y_1, \ldots, y_n)^T \in \mathbb{R}^n\), then \((\alpha(x_1 + y_1), \ldots, \alpha(x_n + y_n))^T = (\alpha x_1 + \alpha y_1, \ldots, \alpha x_n + \alpha y_n)^T\).

(h) If \(\alpha, \beta \in \mathbb{R}, x \in \mathbb{R}^n\), then \(((\alpha + \beta)x_1, \ldots, (\alpha + \beta)x_n)^T = (\alpha x_1 + \beta x_1, \ldots, \alpha x_n + \beta x_n)^T\).

(i) If \(\alpha, \beta \in \mathbb{R}, x \in \mathbb{R}^n\), then \(((\alpha \beta)x_1, \ldots, (\alpha \beta)x_n)^T = (\alpha(\beta x_1), \ldots, \alpha(\beta x_n))^T\).

(j) If \((x_1, \ldots, x_n) \in \mathbb{R}^n\), then \((1x_1, \ldots, 1x_n)^T = (x_1, \ldots, x_n)^T\).

1.13. If \(A\) is \(m \times n\) and \(x, y \in \mathbb{R}^n\), then

\[
A(x + y) = [a_1, a_2, \ldots, a_n] \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} = (x_1 + y_1)a_1 + (x_2 + y_2)a_2 + \cdots + (x_n + y_n)a_n = (x_1a_1 + x_2a_2 + \cdots + x_na_n) + (y_1a_1 + y_2a_2 + \cdots + y_na_n)
\]

\[
= [a_1a_2 \ldots a_n] \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + [a_1a_2 \ldots a_n] \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = Ax + Ay.
\]

If \(A\) is \(m \times n\), \(a \in \mathbb{R}\), and \(x \in \mathbb{R}^n\), then

\[
A(ax) = [a_1a_2 \ldots a_n] \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix} = (\alpha x_1)a_1 + (\alpha x_2)a_2 + \cdots + (\alpha x_n)a_n = \alpha(x_1a_1 + x_2a_2 + \cdots + x_na_n)
\]

\[
= \alpha \begin{pmatrix} a_1a_2 \ldots a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \alpha(Ax).
\]

1.14. \(A + B = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}\)

\(B + A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}\).

1.15. \((A + B) + C = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\)

\(A + (B + C) = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\).
1.16. 
\[(AB)C = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 1 & -4 \end{pmatrix}\]
\[A(BC) = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 1 & -4 \end{pmatrix}.\]

1.17. 
\[A(B + C) = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ -1 & 2 \end{pmatrix}\]
\[AB + AC = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ -1 & 2 \end{pmatrix}.\]

1.18. 
\[(A + B)C = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 0 & 1 \end{pmatrix}\]
\[AC + BC = \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}.\]

1.19. 
\[(\alpha\beta)A = -6 \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -6 & -12 \\ 6 & 0 \end{pmatrix}\]
\[\alpha(\beta A) = 3 \begin{pmatrix} -2 & -4 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -6 & -12 \\ 6 & 0 \end{pmatrix}.\]

1.20. 
\[\alpha(AB) = 3 \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 3 & -3 \end{pmatrix}\]
\[(\alpha A)B = \begin{pmatrix} 3 & 6 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ -3 & -3 \end{pmatrix}\]
\[A(\alpha B) = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -3 & 3 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 3 & -3 \end{pmatrix}.\]

1.21. 
\[(\alpha + \beta)A = 1 \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}\]
\[\alpha A + \beta A = \begin{pmatrix} 3 & 6 \\ -3 & 0 \end{pmatrix} + \begin{pmatrix} -2 & -4 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}.\]

1.22. 
\[\alpha(A + B) = 3 \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 9 \\ 0 & 3 \end{pmatrix}\]
\[\alpha A + \alpha B = \begin{pmatrix} 3 & 6 \\ -3 & 0 \end{pmatrix} + \begin{pmatrix} -3 & 3 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 9 \\ 0 & 3 \end{pmatrix}.\]

1.23. 
\[(A^T)^T = \begin{pmatrix} -2 & 4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 0 & 0 \end{pmatrix} = A.\]

1.24. 
\[\alpha A^T = \begin{pmatrix} 6 & -12 \\ -12 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -12 \\ -12 & 0 \end{pmatrix}\]
\[\alpha A^T = -3 \begin{pmatrix} -2 & 4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -12 \\ -12 & 0 \end{pmatrix}.\]

1.25. 
\[(A + B)^T = \begin{pmatrix} -2 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -1 & 3 \end{pmatrix}\]
\[A^T + B^T = \begin{pmatrix} -2 & 4 \\ 0 & -5 \end{pmatrix} + \begin{pmatrix} -2 & 4 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -1 & 3 \end{pmatrix}.\]
1.26.

\[(AB)^T = \begin{pmatrix} -8 & 22 \\ 0 & -20 \end{pmatrix}^T = \begin{pmatrix} -8 & 0 \\ 22 & -20 \end{pmatrix} \]

\[B^T A^T = \begin{pmatrix} 0 & -2 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} -8 & 0 \\ 22 & -20 \end{pmatrix}.\]

1.27.

\[2x_1 + 3x_2 = 2 \begin{pmatrix} 9 \\ 5 \end{pmatrix} + 3 \begin{pmatrix} -6 \\ -1 \end{pmatrix} = \begin{pmatrix} 18 \\ 10 \end{pmatrix} + \begin{pmatrix} -18 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \]

\[\begin{pmatrix} 9 \\ 5 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \]

1.28.

\[-x_3 + 5x_4 = - \begin{pmatrix} -1 \\ 8 \\ -9 \end{pmatrix} + 5 \begin{pmatrix} 7 \\ -9 \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \end{pmatrix} + \begin{pmatrix} 35 \\ -45 \end{pmatrix} = \begin{pmatrix} 36 \\ -53 \end{pmatrix} \]

\[\begin{pmatrix} -1 \\ 8 \\ -9 \end{pmatrix} \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 36 \\ -53 \end{pmatrix} \]

1.29.

\[4x_2 - 7x_4 - 3x_1 = 4 \begin{pmatrix} -6 \\ -1 \\ 9 \end{pmatrix} - 7 \begin{pmatrix} 7 \\ -9 \\ 5 \end{pmatrix} - 3 \begin{pmatrix} 9 \\ 5 \end{pmatrix} = \begin{pmatrix} -24 \\ -4 \\ -18 \end{pmatrix} + \begin{pmatrix} -49 \\ 63 \end{pmatrix} + \begin{pmatrix} -27 \\ -15 \end{pmatrix} = \begin{pmatrix} -100 \\ -44 \end{pmatrix} \]

\[\begin{pmatrix} -6 \\ -1 \\ -9 \end{pmatrix} \begin{pmatrix} 4 \\ -7 \\ 5 \end{pmatrix} = \begin{pmatrix} -100 \\ -44 \end{pmatrix} \]

1.30.

\[-2v_2 + 4v_3 + 5x_4 = -2 \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ -9 \\ 7 \end{pmatrix} + 5 \begin{pmatrix} 7 \\ -9 \end{pmatrix}. \]

The dimensions do not allow these vectors to be added.

1.31.

\[4v_1 - 3v_4 + 3v_2 + 4v_3 = 4 \begin{pmatrix} 10 \\ -5 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix} = \begin{pmatrix} 40 \\ -20 \\ 12 \end{pmatrix} + \begin{pmatrix} 3 \\ -9 \\ -18 \end{pmatrix} + \begin{pmatrix} 0 \\ 24 \\ 18 \end{pmatrix} + \begin{pmatrix} 0 \\ -36 \\ -28 \end{pmatrix} = \begin{pmatrix} 43 \\ -41 \\ 40 \end{pmatrix} \]
1.32. 
\[ \mathbf{v}_2 - 5\mathbf{v}_1 - 3\mathbf{v}_4 - 2\mathbf{v}_3 = \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} - 5 \begin{pmatrix} 10 \\ -5 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ -9 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} + \begin{pmatrix} -50 \\ 25 \\ -15 \end{pmatrix} + \begin{pmatrix} 3 \\ -9 \\ -18 \end{pmatrix} + \begin{pmatrix} 0 \\ 18 \\ -14 \end{pmatrix} = \begin{pmatrix} -47 \\ 42 \\ -41 \end{pmatrix} \]

1.33. 
\[ A\mathbf{y} = \begin{pmatrix} 9 & -6 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 39 \\ -17 \end{pmatrix} \]

1.34. 
\[ A\mathbf{x}_3 = \begin{pmatrix} 9 & -6 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 8 \end{pmatrix} = \begin{pmatrix} -57 \\ -13 \end{pmatrix} \]

1.35. 
\[ B\mathbf{z} = \begin{pmatrix} 6 & -1 & 7 \\ -1 & 8 & -9 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 20 \\ -17 \end{pmatrix} \]

1.36. 
\[ B\mathbf{w} = \begin{pmatrix} -6 & -1 & 7 \\ -1 & 8 & -9 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} -33 \\ -25 \end{pmatrix} \]

1.37. 
\[ C\mathbf{z} = \begin{pmatrix} 10 & 0 & -1 \\ -5 & 8 & 3 \\ 3 & 6 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -12 \\ 11 \\ 9 \end{pmatrix} \]

1.38. 
\[ C\mathbf{w} = \begin{pmatrix} 10 & 0 & -1 \\ -5 & 8 & 3 \\ 3 & 6 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 23 \\ -19 \\ -12 \end{pmatrix} \]

1.39. 
\[ C\mathbf{v}_3 = \begin{pmatrix} 10 & 0 & -1 \\ -5 & 8 & 3 \\ 3 & 6 & 6 \end{pmatrix} \begin{pmatrix} 0 \\ -9 \\ 7 \end{pmatrix} = \begin{pmatrix} -7 \\ -51 \\ -12 \end{pmatrix} \]

1.40. 
\[ D\mathbf{u} = \begin{pmatrix} 10 & 0 & 0 & -1 \\ -5 & 8 & -9 & 3 \\ 3 & 6 & 7 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -14 \\ 59 \\ 25 \end{pmatrix} \]

1.41. 
\[ \mathbf{y}^T = \begin{pmatrix} 3 \\ -2 \end{pmatrix} = (3 \ -2) \]

1.42. 
\[ \mathbf{u}^T = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix} = (-1 \ 3 \ -2 \ 4) \]

1.43. 
\[ A^T = \begin{pmatrix} 9 & -6 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 9 \ 5 \\ -6 \ -1 \end{pmatrix} \]
7.1. Vectors and Matrices

1.44. 

\[ B^T = \begin{pmatrix} 6 & -1 & 7 \\ -1 & 8 & -9 \end{pmatrix}^T = \begin{pmatrix} 6 & -1 \\ -1 & 8 \\ 7 & -9 \end{pmatrix}. \]

1.45. 

\[ C^T = \begin{pmatrix} 10 & 0 & -1 \\ -5 & 8 & 3 \\ 3 & 6 & 6 \end{pmatrix}^T = \begin{pmatrix} 10 & -5 & 0 \\ -1 & 8 & 6 \\ 0 & 8 & 6 \end{pmatrix}. \]

1.46. 

\[ D^T = \begin{pmatrix} 10 & 0 & 0 & -1 \\ -5 & 8 & -9 & 3 \\ 3 & 6 & 7 & 6 \end{pmatrix}^T = \begin{pmatrix} 10 & -5 & 0 & 0 \\ 0 & 8 & 6 & 0 \\ 0 & -9 & 7 & 0 \\ -1 & 3 & 6 & 3 \end{pmatrix}. \]

1.47. 

\[ 3x + 4y = 7 \text{ is equivalent to } \begin{pmatrix} 3 & 4 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}. \]

1.48. 

\[ -x + 4y = 3 \text{ is equivalent to } \begin{pmatrix} -1 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \]

1.49. 

\[ 3x + 4y = 7 \]
\[ -x + 3y = 2 \]

is equivalent to

\[ \begin{pmatrix} 3 & 4 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}. \]

1.50. 

\[ x - 3y = 5 \]
\[ -2x + 3y = -2 \]

is equivalent to

\[ \begin{pmatrix} 1 & -3 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}. \]

1.51. 

\[ -x_1 + x_3 = 0 \]
\[ 2x_3 + 3x_2 = 3 \]

is equivalent to

\[ \begin{pmatrix} -1 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}. \]

1.52. 

\[ x_1 + 3x_2 - x_3 = 2 \]
\[ -2x_1 + 3x_2 - 2x_3 = -3 \]

is equivalent to

\[ \begin{pmatrix} 1 & 3 & -1 \\ -2 & 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}. \]

1.53. 

\[ -x_1 + x_3 = 0 \]
\[ 2x_1 + 3x_2 = 3 \]
\[ x_2 - x_3 = 4 \]
452 Chapter 7. Matrix Algebra

is equivalent to
\[
\begin{pmatrix}
-1 & 0 & 1 \\
2 & 3 & 0 \\
0 & 1 & -1 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
3 \\
4 \\
\end{pmatrix}.
\]

1.54.

\[
x_1 + 3x_2 - x_3 = 2 \\
-2x_1 + 3x_2 - 2x_3 = -3 \\
2x_1 - x_3 = 0
\]
is equivalent to
\[
\begin{pmatrix}
1 & 3 & -1 & 2 \\
-2 & 3 & -2 & -3 \\
2 & 0 & -1 & 0 \\
\end{pmatrix}.
\]

Section 2. The Geometry of Systems and Linear Equations

2.1. No. There are only three possible solution sets: a point, a line, or no solutions. A circle is none of these.

2.2. The set \( S \) consists of a circle of radius 2, centered at \( x_0 \). The set also includes the center at \( x_0 \). Again, this set is not a point or a line, so it cannot be the solution set of a system of linear equations in two unknowns.

2.3. The set \( S = \{ (t) : t > 0 \} \) in the right half plane in \( \mathbb{R}^2 \), not including the vertical axis. This set is not a point or a line, so it cannot be the solution set of a system of linear equations in two unknowns.

2.4. Assuming that \( s \) is some fixed number, then \( S = \{ (t,s) : t > 0 \} \) is a half line in the plane. This set is not a line or point. It is not infinitely long as we cannot move infinitely in two directions as we can on a line.

2.5. As \( y = t \left( \begin{array}{c} 2 \\ -3 \end{array} \right) \) in a line in \( \mathbb{R}^2 \), one would expect to find a system whose solution set is this line. Indeed, if

\[
\begin{pmatrix}
x \\
y \\
\end{pmatrix} = t \begin{pmatrix}
2 \\
-3 \\
\end{pmatrix},
\]

then \( x = 2t \) and \( y = -3t \). Eliminating the parameter \( t \) by solving one equation for \( t \) and substituting in the other equation. Then, \( t = x/2 \) and

\[
y = -3 \left( \frac{x}{2} \right) \\
2y = -3x \\
3x + 2y = 0.
\]

Of course, there are other systems having this line as their solution set. For example, consider the system

\[
\begin{align*}
3x + 2y &= 0 \\
6x + 4y &= 0.
\end{align*}
\]

2.6. Again,

\[
\begin{pmatrix}
x \\
y \\
\end{pmatrix} = t \begin{pmatrix}
-1 \\
4 \\
\end{pmatrix},
\]

so \( x = -t \) and \( y = 4t \). Eliminating the parameter \( t \) yields \( y = -4x \) or \( 4x + y = 0 \). Other systems will also have this line as a solution. For example, consider

\[
\begin{align*}
4x + y &= 0, \\
8x + 2y &= 0.
\end{align*}
\]

2.7. The line passes through \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) in the direction \( \begin{pmatrix} 2 \\ -3 \end{pmatrix} \). Write

\[
\begin{pmatrix}
x \\
y \\
\end{pmatrix} = \begin{pmatrix}
0 \\
1 \\
\end{pmatrix} + t \begin{pmatrix}
2 \\
-3 \\
\end{pmatrix}.
\]
so that
\[
x = 2t, \\
y = 1 - 3t.
\]

Multiply the first equation by 3 and the second equation by 2.
\[
3x = 6t \\
2y = 2 - 6t.
\]

Add these to get \(3x + 2y = 2\). Of course, other systems are possible. For example, consider
\[
3x + 2y = 2 \\
6x + 4y = 4.
\]

2.8. Again,
\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ -3 \end{pmatrix}
\]
go
ds
\[
x = -2, \\
y = 3 - 3t.
\]

This is somewhat tricky as \(x\) is fixed while \(y\) varies from \(-\infty\) to \(+\infty\). Thus, we are talking about the vertical line with equation \(x = -2\).

2.9. A point has zero dimension. A line is one-dimensional.  

2.10. A point is zero dimensional, a line is one-dimensional, and a plane is two dimensional.  

2.11. In \(\mathbb{R}^4\) we can expect dimensions 0, 1, 2, and 3.  

2.12. Solve for \(x_1\),
\[
x_1 = 2 - 2x_2 + 2x_3 - x_4.
\]

Thus, solutions have the form
\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 - 2x_2 + 2x_3 - x_4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]

The solution set has dimension 3 ((2, 0, 0, 0)\(^T\) is just a translation).  

2.13. Solve the second equation for \(x_2\),
\[
x_2 = 3 + 3x_3 + x_4.
\]

Substitute in the first equation and solve for \(x_1\),
\[
x_1 + 2(3 + 3x_3 + x_4) - 2x_3 + x_4 = 2 \\
x_1 + 6 + 4x_3 + 3x_4 = 2 \\
x_1 = -4 - 4x_3 - 3x_4.
\]

Thus, solutions have the form
\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4 - 4x_3 - 3x_4 \\ 3 + 3x_3 + x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}.
\]

This set has dimension 2. Think of the set as a translated plane in \(\mathbb{R}^4\).  

2.14. Solve the third equation for \(x_3\),
\[
x_3 = x_4.
\]
Substitute this in the second equation and solve for $x_2$

\[
x_2 - 3x_4 - x_4 = 3
\]

\[
x_2 = 3 + 4x_4.
\]

Substitute both of these in the first equation and solve for $x_1$

\[
x_1 + 2(3 + 4x_4) - 2x_4 + x_4 = 2
\]

\[
x_1 + 6 + 7x_4 = 2
\]

\[
x_1 = -4 - 7x_4.
\]

Thus, solutions have the form

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  -4 - 7x_4 \\
  3 + 4x_4 \\
  3 \\
  0
\end{pmatrix} + x_4 \begin{pmatrix}
  4 \\
  0 \\
  1 \\
  1
\end{pmatrix}.
\]

This solution has dimension 1. Think of the set as a translated line in $\mathbb{R}^4$.

### Section 3. Solving Systems of Equations

#### 3.1. Solve for $x$.

\[
2x - 3y = 0
\]

\[
2x = 3y
\]

\[
x = \frac{3}{2}y
\]

Thus, the solution can be written

\[
x = \begin{pmatrix}
  x \\
  y
\end{pmatrix} = \begin{pmatrix}
  \frac{3}{2}y \\
  y
\end{pmatrix} = y \begin{pmatrix}
  \frac{3}{2} \\
  1
\end{pmatrix},
\]

where $y$ is free. Thus, all solutions are scalar multiples of $(3/2, 1)^T$ and lie on the line through the origin with direction vector $(3/2, 1)^T$.

#### 3.2. Solve for $x$.

\[
-2x + 4y = 0
\]

\[
-2x = -4y
\]

\[
x = 2y
\]

Thus, the solutions can be written

\[
x = \begin{pmatrix}
  x \\
  y
\end{pmatrix} = \begin{pmatrix}
  2y \\
  y
\end{pmatrix} = y \begin{pmatrix}
  2 \\
  1
\end{pmatrix},
\]

where $y$ is free. Alternatively, you can let $y = \alpha$ and write $x = \alpha(2, 1)^T$, where $\alpha$ is any real number.

#### 3.3. We solve for $x = (5 + 3y)/2$. Hence the solution space consists of all vectors of the form

\[
x = \begin{pmatrix}
  x \\
  y
\end{pmatrix} = \begin{pmatrix}
  (5 + 3y)/2 \\
  y
\end{pmatrix} = \begin{pmatrix}
  5/2 \\
  0
\end{pmatrix} + y \begin{pmatrix}
  3/2 \\
  1
\end{pmatrix},
\]

where $y$ is an arbitrary real number.

#### 3.4. Solve for $x$.

\[
-2x + 4y = 7
\]

\[
-2x = 7 - 4y
\]

\[
x = -\frac{7}{2} + 2y
\]

The solution is

\[
x = \begin{pmatrix}
  x \\
  y
\end{pmatrix} = \begin{pmatrix}
  -7/2 + 2y \\
  y
\end{pmatrix} = \begin{pmatrix}
  -7/2 \\
  0
\end{pmatrix} + y \begin{pmatrix}
  2 \\
  1
\end{pmatrix},
\]
where \( y \) is free. Alternatively, you can select a different parameter and write \( x = (-7/2, 0)^T + \alpha(2, 1)^T \), where \( \alpha \) is any real number.

3.5. In matrix form the system is

\[
\begin{pmatrix} 2 & -3 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.
\]

The augmented matrix is

\[
M = \begin{pmatrix} 2 & -3 & 0 \\ 1 & -4 & 0 \end{pmatrix}
\]

We reduce to row echelon form by first interchanging the rows and then subtracting 2 times the first row from the second.

\[
M \rightarrow \begin{pmatrix} 1 & -4 & 2 \\ 2 & -3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -4 & 2 \\ 0 & 5 & -4 \end{pmatrix}
\]

The simplified system of equations is

\[
x - 4y = 2 \\
5y = -4.
\]

Backsolving we get \( y = -4/5 \) and then \( x = 2 + 4y = 2 - 16/5 = -6/5 \). Hence the solution is \((-6/5, -4/5)^T\).

3.6. Set up the augmented matrix.

\[
\begin{pmatrix} -2 & 4 & 0 \\ -1 & 1 & 4 \end{pmatrix}
\]

Multiply row 1 by \(-1/2\) and add to row 2.

\[
\begin{pmatrix} -2 & 4 & 0 \\ 0 & -1 & 4 \end{pmatrix}
\]

This gives an equivalent system

\[
-2x + 4y = 0 \\
-2x = 16,
\]

The second equation gives \( y = -4 \). Substitute this in the first equation to get

\[
-2x + 4(-4) = 0, \\
x = -8.
\]

The solution is \((-8, -4)\).

3.7. In matrix form the system is

\[
\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}.
\]

The augmented matrix is

\[
M = \begin{pmatrix} 2 & -3 & 5 \\ 1 & 4 & 7 \end{pmatrix}
\]

We reduce to row echelon form by first interchanging the rows and then subtracting 2 times the first row from the second.

\[
M \rightarrow \begin{pmatrix} 1 & 4 & 7 \\ 2 & -3 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & 7 \\ 0 & -11 & -9 \end{pmatrix}
\]

The simplified system of equations is

\[
x + 4y = 7 \\
-11y = -9.
\]

Backsolving we get \( y = 9/11 \) and then \( x = 7 - 4y = 7 - 36/11 = 41/11 \). Hence the solution is \((41/11, 9/11)^T\).

3.8. Set up the augmented matrix.

\[
\begin{pmatrix} -2 & 4 & 7 \\ -3 & -4 & 0 \end{pmatrix}
\]
Multiply row 1 by 3/2 and add to row 2.

\[
\begin{pmatrix}
-2 & 4 & 7 \\
0 & 2 & 21/2
\end{pmatrix}
\]

This gives

\[-2x + 4y = 7,\]

\[2y = \frac{21}{2}.\]

The second equation gives \(y = 21/4\). Substitute this in the first equation.

\[-2x + 4 \left( \frac{21}{4} \right) = 7\]

\[-2x + 21 = 7\]

\[-2x = -14\]

\[x = 7\]

The solution is \((7, 21/4)^T\).

3.9. Set up the augmented matrix

\[
\begin{pmatrix}
1 & -4 & 19 \\
8 & 13 & -28
\end{pmatrix}
\]

Multiply row 1 by \(-8\) and add to row 2.

\[
\begin{pmatrix}
1 & -4 & 19 \\
0 & 45 & -180
\end{pmatrix}
\]

This gives

\[x - 4y = 19,\]

\[45y = -180.\]

The second equation gives \(y = -4\). Substitute this in the first equation.

\[x - 4(-4) = 19\]

\[x + 16 = 19\]

\[x = 3\]

The solutions is \((3, -4)^T\).

3.10. Set up the augmented matrix

\[
\begin{pmatrix}
18 & -12 & 96 \\
8 & -2 & 36
\end{pmatrix}
\]

Multiply the first row by \(-8/18\) (or \(-2/9\)) and add to the second row.

\[
\begin{pmatrix}
18 & -12 & 96 \\
0 & 10/3 & -20/3
\end{pmatrix}
\]

This gives

\[18x - 12y = 96,\]

\[\frac{10}{3}y = -\frac{20}{3}.\]

The second equation gives \(y = -2\). Substitute this in the first equation.

\[18x - 12(-2) = 96\]

\[18x + 24 = 96\]

\[18x = 72\]

\[x = 4\]

The solution is \((4, -2)^T\).
3.11. Set up the augmented matrix.
\[
\begin{pmatrix}
2 & 3 & 5 \\
4 & 6 & 7
\end{pmatrix}
\]
Multiply the first row by $-2$ and add to row 2.
\[
\begin{pmatrix}
2 & 3 & 5 \\
0 & 0 & -3
\end{pmatrix}
\]
The last row represents the equation $0x + 0y = -3$, which has no solutions. Therefore, the system is inconsistent.

3.12. Set up the augmented matrix.
\[
\begin{pmatrix}
-2 & 4 & 7 \\
4 & -8 & 10
\end{pmatrix}
\]
Multiply row 1 by 2 and add to row 2.
\[
\begin{pmatrix}
-2 & 4 & 7 \\
0 & 0 & 24
\end{pmatrix}
\]
The last row represents the equation $0x + 0y = 24$, which has no solutions. Therefore, the system has no solutions.

3.13. In matrix form the system is
\[
\begin{pmatrix}
4 & 7 & 5 \\
-2 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
x \\ y \\ z
\end{pmatrix} = \begin{pmatrix}
18 \\ 0
\end{pmatrix}
\]
The augmented matrix is
\[
M = \begin{pmatrix}
4 & 7 & 5 & 18 \\
-2 & 1 & -1 & 0
\end{pmatrix}
\]
We reduce to row echelon form by first interchanging the rows and then subtracting 2 times the first row from the second:
\[
M \rightarrow \begin{pmatrix}
-2 & 1 & -1 & 0 \\
4 & 7 & 5 & 18
\end{pmatrix} \rightarrow \begin{pmatrix}
-2 & 1 & -1 & 0 \\
0 & 9 & 3 & 18
\end{pmatrix}
\]
The simplified system is
\[-2x + y - z = 0 \\
9y + 3z = 18
\]
To backsolve we first set the free variable $z = t$. Next we solve for $y = 2 - t/3$. Finally we solve for $x = (y - z)/2 = 1 - 2t/3$. Hence the solutions are the vectors of the form
\[
x = \begin{pmatrix}
x \\ y \\ z
\end{pmatrix} = \begin{pmatrix}
1 - 2t/3 \\ 2 - t/3 \\ t
\end{pmatrix} + t \begin{pmatrix}
-2/3 \\ -1/3 \\ 1
\end{pmatrix}
\]

3.14. In matrix form the system is
\[
\begin{pmatrix}
10 & -5 & 3 \\
-20 & 18 & 0
\end{pmatrix}
\begin{pmatrix}
x \\ y \\ z
\end{pmatrix} = \begin{pmatrix}
2 \\ -2
\end{pmatrix}
\]
The augmented matrix is
\[
M = \begin{pmatrix}
10 & -5 & 3 & 2 \\
-20 & 18 & 0 & -2
\end{pmatrix}
\]
We reduce to row echelon form by adding 2 times the first row to the second:
\[
M \rightarrow \begin{pmatrix}
10 & -5 & 3 & 2 \\
0 & 8 & 6 & 2
\end{pmatrix}
\]
The simplified system is
\[10x - 5y + 3z = 2 \\
8y + 6z = 2
\]
To backsolve, we first set the free variable \( z = t \). Then \( y = (2 - 6z)/8 = (2 - 6t)/8 \). Finally, \( x = 2 + 5y - 3z = 2 + 5(2 - 6t)/8 - 3t = 13/4 - 27t/4 \). Hence the solutions are the vectors

\[
\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13/4 - 27t/4 \\ 1/4 - 3t/4 \\ t \end{pmatrix} = \begin{pmatrix} 13/4 \\ 1/4 \\ 0 \end{pmatrix} + t \begin{pmatrix} -27/4 \\ -3/4 \\ 1 \end{pmatrix}.
\]

3.15. Set up the augmented matrix.

\[
\begin{pmatrix} 0 & -9 & 7 & -32 \\ -1 & 30 & -15 & 89 \end{pmatrix}
\]

Swap rows 1 and 2.

\[
\begin{pmatrix} -1 & 30 & -15 & 89 \\ 0 & -9 & 7 & -32 \end{pmatrix}
\]

This gives

\[-x + 30y - 15z = 89,
-9y + 7z = -32.\]

Thus, \( x \) and \( y \) are pivot variables and \( z \) is free. Solve the second equation for \( y \).

\[-9y = -32 - 7z,\]
\[y = \frac{32}{9} + \frac{7}{9}z.\]

Substitute this in the first equation and solve for \( x \).

\[-x + 30 \left( \frac{32}{9} + \frac{7}{9}z \right) - 15z = 89,
-x + \frac{320}{3} + \frac{70}{3} - \frac{45}{3}z = \frac{267}{3},
-x = -\frac{53}{3} - \frac{25}{3}z,
-x = \frac{53}{3} + \frac{25}{3}z,
\]

Thus,

\[
\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 53/3 + (25/3)z \\ 32/9 + (7/9)z \end{pmatrix} = \begin{pmatrix} 53/3 \\ 32/9 \end{pmatrix} + z \begin{pmatrix} 25/3 \\ 7/9 \end{pmatrix},
\]

where \( z \) is free.

3.16. Set up the augmented matrix.

\[
\begin{pmatrix} -3 & -6 & -3 & -3 \\ -6 & 4 & 1 & -8 \end{pmatrix}
\]

Multiply row 1 by \(-2\) and add to row 2.

\[
\begin{pmatrix} -3 & -6 & -3 & -3 \\ 0 & 16 & 7 & -2 \end{pmatrix}
\]

This gives

\[-3x - 6y - 3z = -3,
16y + 7z = -2.\]

Thus, \( x \) and \( y \) are pivot variables and \( z \) is free. Solve the second equation for \( y \).

\[16y = -2 + 7z,
\]
\[y = \frac{-1}{8} + \frac{7}{16}z.\]
Substitute this result in the first equation and solve for $x$.

\[
-3x - 6 \left( -\frac{1}{8} + \frac{7}{16}z \right) - 3z = -3 \\
-3x + \frac{3}{4} \frac{21}{8}z - \frac{24}{8}z = -\frac{12}{4} \\
-3x = -\frac{15}{4} + \frac{45}{8}z \\
x = \frac{5}{4} - \frac{15}{8}z
\]

Thus,

\[
x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5/4 - 15/8z \\ -1/8 + 7/16z \\ z \end{pmatrix} = \begin{pmatrix} 5/4 \\ -1/8 \\ 0 \end{pmatrix} + z \begin{pmatrix} -15/8 \\ 7/16 \\ 1 \end{pmatrix},
\]

where $z$ is free.

3.17. Set up the augmented matrix and reduce.

\[
\begin{pmatrix} -6 & 8 & 0 & 2 \\
4 & 8 & 8 & 20 \\
-2 & 2 & 7 & 7 \\
\end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
\end{pmatrix}
\]

This gives $x = (1, 1, 1)^T$.

3.18. Set up the augmented matrix and reduce.

\[
\begin{pmatrix} 3 & -3 & 1 & 0 \\
7 & -4 & 5 & 3 \\
4 & -3 & -3 & 1 \\
\end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

Thus, $x = (1, 1, 0)^T$.

3.19. The augmented matrix is

\[
M = \begin{pmatrix} 4 & 2 & -5 & -5 \\
-14 & -8 & 18 & 16 \\
-3 & -2 & 4 & 3 \\
\end{pmatrix}
\]

Using a computer we reduce this to row echelon form

\[
\begin{pmatrix} 4 & 2 & -5 & -5 \\
0 & -1 & 1/2 & -3/2 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

The simplified system is

\[
4y_1 + 2y_2 - 5y_3 = -5 \\
-2y_2 + y_3 = -3/2.
\]

To backsolve, we first set the free variable $y_3 = t$. Next we solve for $y_2 = 3/2 + y_3/2 = (3 + t)/2$. Finally, we solve for $y_1 = (-5 - 2y_2 + 5y_3)/4 = [-5 - (3 + t) + 5t]/4 = -2 + t$. Hence our solutions are all vectors of the form

\[
y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -2 + t \\ (3 + t)/2 \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 3/2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix}.
\]

3.20. Set up the augmented matrix and reduce.

\[
\begin{pmatrix} -4 & 10 & -6 & -14 \\
0 & -4 & 4 & 4 \\
2 & -10 & 8 & 12 \\
\end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 \\
0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

This gives

\[
x - z = 1, \\
y - z = -1.
\]
Thus, $x$ and $y$ are pivot variables and $z$ is free. Solve each equation for its pivot variable.

\[
x = 1 + z \\
y = -1 + z
\]

Thus,

\[
x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 + z \\ -1 + z \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]

where $z$ is free.

3.21. The augmented matrix is

\[
M = \begin{pmatrix} -3 & -3 & 1 & 4 \\ 8 & 7 & -2 & -8 \\ 8 & 6 & -1 & -5 \end{pmatrix}
\]

Using a computer we reduce this to the row echelon form

\[
\begin{pmatrix} -3 & -3 & 1 & 4 \\ 0 & -1 & 2/3 & 8/3 \\ 0 & 0 & 1/3 & 1/3 \end{pmatrix}
\]

the simplified system is

\[
-3y_1 - 3y_2 + y_3 = 4 \\
-y_2 + 2y_3/3 = 8/3 \\
y_3/3 = 1/3.
\]

Backsolving we get $y_3 = 1$. Next $y_2 = -8/3 + 2/3 = -2$. Finally, $y_1 = (-4 - 3y_2 + y_3)/3 = 1$. Hence the only solution is $y = (1, -2, 1)^T$.

3.22. Set up the augmented matrix and reduce.

\[
\begin{pmatrix} 5 & 9 & 2 & 8 \\ -2 & -3 & -1 & -2 \\ 0 & -2 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

Thus, $x = (-2, 2, 0)^T$.

3.23. Set up the augmented matrix and reduce.

\[
\begin{pmatrix} -12 & 12 & -8 & -8 \\ -16 & 16 & -10 & -10 \\ -3 & 3 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

This gives

\[
x - y = 0, \\
z = 1.
\]

Thus, $x$ and $z$ are pivot variables and $y$ is free. Solve each equation for its pivot variable.

\[
x = y \\
z = 1
\]

Thus

\[
x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]

where $y$ is free.

3.24. Set up the augmented matrix and reduce.

\[
\begin{pmatrix} 0 & 4 & 6 & -7 & 4 \\ -4 & 10 & 4 & -8 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 11/4 & -19/8 & 1 \\ 0 & 1 & 3/2 & -7/4 & 1 \end{pmatrix}
\]
Thus,
\[
\begin{align*}
x_1 + \frac{11}{4}x_3 - \frac{19}{8}x_4 &= 1, \\
x_2 + \frac{3}{2}x_3 - \frac{7}{4}x_4 &= 1.
\end{align*}
\]
Thus, \(x_1\) and \(x_2\) are pivot variables, while \(x_3\) and \(x_4\) are free. Solve each equation for its pivot variable.
\[
\begin{align*}
x_1 &= 1 - \frac{11}{4}x_3 + \frac{19}{8}x_4, \\
x_2 &= 1 - \frac{3}{2}x_3 + \frac{7}{4}x_4.
\end{align*}
\]
Thus,
\[
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 - (11/4)x_3 + (19/8)x_4 \\ 1 - (3/2)x_3 + (7/4)x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -11/4 \\ -3/2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 19/8 \\ 7/4 \\ 0 \\ 1 \end{pmatrix},
\]
where \(x_3\) and \(x_4\) are free.

3.25. Set up the augmented matrix and reduce.
\[
\begin{pmatrix}
-5 & -4 & 4 & 3 & 17 \\
7 & 6 & -5 & 3 & 12
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & -2 & -15 & -75 \\
0 & 1 & 3/2 & 18 & 179/2
\end{pmatrix}
\]
This gives
\[
\begin{align*}
x_1 - 2x_3 - 15x_4 &= -75, \\
x_2 + \frac{3}{2}x_3 + 18x_4 &= \frac{179}{2}.
\end{align*}
\]
Thus, \(x_1\) and \(x_2\) are pivot variables, while \(x_3\) and \(x_4\) are free. Solve each equation for its pivot variable.
\[
\begin{align*}
x_1 &= -75 + 2x_3 + 15x_4, \\
x_2 &= \frac{179}{2} - \frac{3}{2}x_3 - 18x_4.
\end{align*}
\]
Thus,
\[
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -75 + 2x_3 + 15x_4 \\ 179/2 - (3/2)x_3 - 18x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -75 \\ 179/2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ -3/2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 15 \\ -18 \\ 0 \\ 1 \end{pmatrix},
\]
where \(x_3\) and \(x_4\) are free.

3.26. Set up the augmented matrix and reduce.
\[
\begin{pmatrix}
2 & -3 & -2 & 2 & -4 \\
-4 & 4 & 0 & 3 & -7 \\
8 & -8 & -1 & -7 & 13
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 0 & -25/4 & 29/4 \\
0 & 1 & 0 & -11/2 & 11/2 \\
0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]
This gives
\[ x_1 - \frac{25}{4} x_4 = \frac{29}{4}, \]
\[ x_2 - \frac{11}{2} x_4 = \frac{11}{2}, \]
\[ x_3 + x_4 = 1. \]

Thus, \( x_1, x_2, \) and \( x_3 \) are pivot variables and \( x_4 \) is free. Solve each equation for its pivot variable.
\[ x_1 = \frac{29}{4} \frac{25}{4} x_4 \]
\[ x_2 = \frac{11}{2} + \frac{11}{2} x_4 \]
\[ x_3 = 1 - x_4 \]

Thus,
\[ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{29}{4} + \frac{25}{4} x_4 \\ \frac{11}{2} + \frac{11}{2} x_4 \\ 1 - x_4 \\ x_4 \end{pmatrix}. \]

where \( x_4 \) is free.

3.27. The augmented matrix is
\[ M = \begin{pmatrix} -7 & 7 & -8 & -3 & 37 \\ 9 & -5 & 8 & -2 & -35 \\ 5 & 0 & 2 & 8 & -9 \end{pmatrix}. \]

Using a computer to perform row operations this is reduced to the row echelon form
\[ \begin{pmatrix} -7 & 7 & -8 & -3 & 37 \\ 0 & 4 & -16/7 & -41/7 & 88/7 \\ 0 & 0 & -6/7 & 369/28 & 12/7 \end{pmatrix} \]

The simplified system of equations is
\[ -7y_1 + 7y_2 - 8y_3 - 3y_4 = 37 \]
\[ 4y_2 - 16y_3/7 - 41y_4/7 = 88/7 \]
\[ -6y_3/7 + 369y_4/28 = 12/7 \]

To backsolve, we first set the free variable \( y_4 = t \). Then we solve for \( y_3 = (-12 + 369t/4)/6 = -2 + 123t/8 \).

Next,
\[ y_2 = (88/7 + 16y_3/7 + 41y_4/7)/4 \]
\[ = [88 + 16(-2 + 123t/8) + 41t]/28 \]
\[ = [56 + 287t]/28 \]
\[ = 2 + 41t/4. \]

Finally
\[ y_1 = [-37 + 7y_2 - 8y_3 - 3y_4]/7 \]
\[ = [-37 + 7(2 + 41t/4) - 8(-2 + 123t/8) - 3t]/7 \]
\[ = -1 - 31t/4. \]

Hence the solutions are the vectors
\[ y = \begin{pmatrix} -1 - 31t/4 \\ 2 + 41t/4 \\ -2 + 123t/8 \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -31/4 \\ 41/4 \\ 123/8 \\ 1 \end{pmatrix}. \]
3.28. Set up the augmented matrix and reduce.

\[
\begin{pmatrix}
-7 & -4 & -5 & -9 & 31 \\
1 & 10 & 7 & 10 & -18 \\
-2 & 0 & -3 & 0 & -2
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 & 75/55 & -379/77 \\
0 & 1 & 0 & 19/14 & -57/14 \\
0 & 0 & 1 & -50/77 & 304/77
\end{pmatrix}
\]

This gives

\[
x_1 + \frac{75}{77}x_4 = -\frac{379}{77},
\]
\[
x_2 + \frac{19}{14}x_4 = -\frac{57}{14},
\]
\[
x_3 - \frac{50}{77}x_4 = \frac{304}{77}.
\]

Therefore, \(x_1, x_2,\) and \(x_3\) are pivot variables and \(x_4\) is free. Solve each equation for its pivot variable.

\[
x_1 = -\frac{379}{77} - \frac{75}{77}x_4
\]
\[
x_2 = -\frac{57}{14} - \frac{19}{14}x_4
\]
\[
x_3 = \frac{304}{77} + \frac{50}{77}x_4
\]

Thus,

\[
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{379}{77} - (75/77)x_4 \\ -\frac{57}{14} - (19/14)x_4 \\ \frac{304}{77} + (50/77)x_4 \\ x_4 \end{pmatrix} + x_4 \begin{pmatrix} -75/77 \\ -19/14 \\ 50/77 \\ 1 \end{pmatrix},
\]

where \(x_4\) is free.

3.29. The augmented matrix is

\[
M = \begin{pmatrix}
8 & -6 & 9 & 8 & -1 & 15 \\
-9 & 5 & -7 & 9 & 0 & -30 \\
1 & -4 & 1 & -3 & -7 & 9
\end{pmatrix}
\]

With a computer we reduce this to row echelon form

\[
\begin{pmatrix}
8 & -6 & 9 & 8 & -1 & 15 \\
0 & -14 & 25 & 144 & -9 & -105 \\
0 & 0 & -83 & -524 & -67 & 441
\end{pmatrix}
\]

The simplified system of equation is

\[
8y_1 - 6y_2 + 9y_3 + 8y_4 - y_5 = 15
\]
\[
-14y_2 + 25y_3 + 144y_4 - 9y_5 = -105
\]
\[
-83y_3 - 524y_4 - 67y_5 = 441
\]

To backsolve, we first set the free variables \(y_4 = s\) and \(y_5 = t\). The we solve for

\[
y_3 = [-441 - 524s - 67t]/83.
\]

Next,

\[
y_2 = [105 + 25y_3 + 144y_4 - 9y_3]/14
\]
\[
= [105 + 25(-441 - 524s - 67t)/83] = 144s - 9t]/14
\]
\[
= [-165 - 82s - 173t]/83.
\]
and finally,
\[ y_1 = \frac{15 + 6y_2 - 9y_3 - 8y_4 + y_5}{8} = \frac{15 + 6(-165 + 82s + 173t)/83 - 9(-441 - 524s - 67t)/83 - 8s + t}{8} = \frac{528 + 445s - 44t}{83}. \]

Hence the solutions are the vectors

\[
\begin{pmatrix}
\frac{528 + 445s - 44t}{83} \\
\frac{-165 - 82s - 173t}{83} \\
\frac{-441 - 524s - 67t}{83}
\end{pmatrix}
= \begin{pmatrix} s \\ t \end{pmatrix}
= \begin{pmatrix}
\frac{528}{83} \\
\frac{-165}{83} \\
\frac{-441}{83}
\end{pmatrix} + \begin{pmatrix}
\frac{445}{83} \\
\frac{-82}{83} \\
\frac{-524}{83}
\end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}
+ \begin{pmatrix}
\frac{-44}{83} \\
\frac{-173}{83} \\
\frac{-67}{83}
\end{pmatrix}. 
\]

3.30. The augmented matrix is

\[
M = \begin{pmatrix}
-2 & 3 & 6 & -7 & -1 & 3 \\
-1 & -6 & 1 & -6 & -8 & 1 \\
0 & -9 & -5 & -1 & 9 & 16
\end{pmatrix}. 
\]

With a computer we reduce this to row echelon form

\[
\begin{pmatrix}
-2 & 3 & 6 & -7 & -1 & 3 \\
0 & 15 & 4 & 5 & 15 & 1 \\
0 & 0 & 13 & -10 & -90 & -83
\end{pmatrix}. 
\]

The simplified system of equations is

\[
\begin{align*}
-2y_1 + 3y_2 + 6y_3 - 7y_4 - y_5 &= 3 \\
15y_2 + 4y_3 + 5y_4 + 15y_5 &= 1 \\
13y_3 - 10y_4 - 90y_5 &= -83 \\
\end{align*}
\]

To backsolve, we first set the free variables \( y_4 = s \) and \( y_5 = t \). Then we solve for

\[ y_3 = \frac{-83 + 10s + 90t}{13}. \]

Next,

\[
\begin{align*}
y_2 &= \frac{1 - 4y_3 - 5y_4 - 15y_5}{15} \\
&= \frac{1 - 4(-83 + 10s + 90t)/13 - 5s - 15t}{15} \\
&= \frac{23 - 7s - 37t}{13}. \\
\end{align*}
\]

and finally,

\[
\begin{align*}
y_1 &= \frac{3 - 3y_2 - 6y_3 + 7y_4 + y_5}{2} \\
&= \frac{3 - 3(23 - 7s - 37t)/13 - 6(-83 + 10s + 90t)/13 + 7s + t}{2} \\
&= -18 - 2s + 16t. \\
\end{align*}
\]
Hence the solutions are the vectors

\[
y = \begin{pmatrix}
-18 - 2s + 16t \\
[23 - 7s - 37t]/13 \\
[-83 + 10s + 90t]/13 \\
s \\
t
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-18 \\
23/13 \\
-83/13 \\
0 \\
0
\end{pmatrix}
+ s \begin{pmatrix}
-2 \\
-7/13 \\
+10/13 \\
1 \\
0
\end{pmatrix}
+ t \begin{pmatrix}
16 \\
-37/13 \\
90/13 \\
0 \\
1
\end{pmatrix}.
\]

Section 4. Properties of Solution Sets

4.1. Consistent
4.2. Inconsistent
4.3. Inconsistent
4.4. Inconsistent
4.5. Consistent
4.6. Consistent
4.7. Using row operations we reduce \( A \) to row echelon form.

\[
A \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.
\]

We now see that the only solution of the homogeneous system is the zero vector \((0, 0)^T\). Hence the matrix is nonsingular.

4.8. The augmented matrix reduces to

\[
\begin{pmatrix} 1 & -2 & 0 \\ 2 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The solutions are

\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix},
\]

where \( x_2 \) is free. The coefficient matrix is singular.

4.9. The augmented matrix reduces to

\[
\begin{pmatrix} -1 & 1 & 0 \\ 3 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The solutions are

\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]

where \( x_2 \) is free. The coefficient matrix is singular.

4.10. The augmented matrix reduces to

\[
\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

The unique solution is \( x = (0, 0)^T \). The coefficient matrix is nonsingular.

4.11. Using row operations we reduce \( A \) to row echelon form

\[
A \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

We now see that the only solution of the homogeneous system is the zero vector \((0, 0)^T\). Hence the matrix is nonsingular.
4.12. The augmented matrix reduces to
\[
\begin{pmatrix}
1 & 1 & 2 & 0 \\
1 & 0 & 1 & 0 \\
1 & 1 & 2 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
The solutions are
\[
x = \begin{pmatrix}
-x_3 \\
-x_3 \\
x_3
\end{pmatrix}
= x_3 \begin{pmatrix}
-1 \\
-1 \\
1
\end{pmatrix},
\]
where \(x_3\) is free. The coefficient matrix is singular.

4.13. The augmented matrix reduces to
\[
\begin{pmatrix}
-1 & 1 & 1 & 0 \\
1 & 0 & 2 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 3 & 0
\end{pmatrix}
\]
The solutions are
\[
x = \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= \begin{pmatrix}
-2x_3 \\
-3x_3 \\
x_3
\end{pmatrix}
= x_3 \begin{pmatrix}
-2 \\
-3 \\
1
\end{pmatrix},
\]
where \(x_3\) is free. The coefficient matrix is singular.

4.14. The augmented matrix reduces to
\[
\begin{pmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
The solutions are
\[
x = \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= \begin{pmatrix}
0 \\
-x_3 \\
x_3
\end{pmatrix}
= x_3 \begin{pmatrix}
0 \\
-1 \\
1
\end{pmatrix},
\]
where \(x_3\) is free. The coefficient matrix is singular.

4.15. The augmented matrix \([A\ I]\) reduces to
\[
\begin{pmatrix}
1 & 2 & 1 & 0 \\
0 & 4 & 0 & 1
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 1 & -1/2 \\
0 & 1 & 0 & 1/4
\end{pmatrix}
\]
Thus, matrix \(A\) is nonsingular and
\[
A^{-1} = \begin{pmatrix}
1 & -1/2 \\
0 & 1/4
\end{pmatrix}.
\]

4.16. The augmented matrix \([A\ I]\) reduces to
\[
\begin{pmatrix}
3 & -1 & 1 & 0 \\
0 & -1 & 1 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & -1/3 & 1/3 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
Matrix \(A\) is singular and has no inverse.

4.17. The augmented matrix \([A\ I]\) reduces to
\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]
Matrix \(A\) is singular and has no inverse.

4.18. We augment \(A\) with the identity and perform row operations:
\[
\begin{pmatrix}
0 & -4 & 1 & 0 \\
-1 & 2 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -2 & 0 & -1 \\
0 & 1 & -1/4 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & -1/2 & -1 \\
0 & 1 & -1/4 & 0
\end{pmatrix}
\]
Hence \(A\) is nonsingular and
\[
A^{-1} = \begin{pmatrix}
-1/2 & -1 \\
-1/4 & 0
\end{pmatrix}.
\]
4.19. The augmented matrix \([A I]\) reduces to
\[
\begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\] 
Matrix \(A\) is singular and has no inverse.

4.20. The augmented matrix \([A I]\) reduces to
\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\] 
Thus, matrix \(A\) is nonsingular and
\[
A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.
\]

4.21. We augment \(A\) with the identity and perform row operations:
\[
[A, I] \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}
\]
Hence \(A\) is nonsingular and
\[
A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1/2 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}.
\]

4.22. The augmented matrix \([A I]\) reduces to
\[
\begin{bmatrix}
1 & 2 & -3 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & -5 & 1 & 0 & -2 \\
0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\] 
Matrix \(A\) is singular and has no inverse.

4.23. No. There are fewer equations than unknowns, so there are free variables. There is either no solution or there are many solutions.

4.24. The system
\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3
\end{align*}
\]
has more unknowns than equations. Therefore, there must be free variables. The system is either inconsistent or has an infinite solution set.

4.25. The coefficient matrix, in row echelon form, is
\[
\begin{pmatrix}
1 & 2 \\
1 & -1
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 2 \\
0 & -3
\end{pmatrix}.
\]
The row echelon form has nonzero entries on its diagonal, so the coefficient matrix is nonsingular and the system has a unique solution.

4.26. The coefficient matrix, in row echelon form, is
\[
\begin{pmatrix}
1 & 2 \\
2 & 4
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 2 \\
0 & 0
\end{pmatrix}.
\]
The presence of a zero diagonal element in the row echelon form indicates that the coefficient matrix is singular. The system does not have a unique solution.

4.27. In matrix form,
\[
\begin{pmatrix}
-1 & -3 \\
2 & 6
\end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}.
\]
and the coefficient matrix has row echelon form
\[
\begin{pmatrix}
-1 & -3 \\
2 & 6
\end{pmatrix} \sim \begin{pmatrix}
-1 & -1 \\
0 & 0
\end{pmatrix}.
\]
The presence of a zero on the diagonal of the row echelon form indicates that the coefficient matrix is singular. The system does not have a unique solution.

4.28. In matrix form,
\[
\begin{pmatrix}
1 & 2 \\
2 & 1
\end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]
and the coefficient matrix has row echelon form
\[
\begin{pmatrix}
1 & 2 \\
2 & 1
\end{pmatrix} \sim \begin{pmatrix}
1 & 2 \\
0 & -3
\end{pmatrix}.
\]
All diagonal elements in the row echelon form are nonzero, so the coefficient matrix is nonsingular and the system has a unique solution.

4.29. The coefficient matrix has row echelon form
\[
\begin{pmatrix}
1 & 1 & 2 \\
1 & 1 & -1 \\
1 & -2 & 2
\end{pmatrix} \sim \begin{pmatrix}
1 & 1 & 2 \\
0 & 1 & -3 \\
0 & 0 & -3
\end{pmatrix}.
\]
All diagonal elements in the row echelon form are nonzero, so the coefficient matrix is nonsingular and the system has a unique solution.

4.30. The coefficient matrix has row echelon form
\[
\begin{pmatrix}
1 & 0 & 3 \\
-1 & 1 & -1 \\
0 & 2 & 4
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 3 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{pmatrix}.
\]
The presence of a zero on the diagonal of the row echelon form indicates that the coefficient matrix in singular. The system does not have a unique solution.

4.31. The augmented matrix is
\[
\begin{pmatrix}
0 & 2 & -4 & 0 \\
3 & -5 & 10 & a \\
2 & -4 & 8 & b
\end{pmatrix}.
\]
We use row operations to reduce this to row echelon form
\[
\begin{pmatrix}
1 & -2 & 4 & b/2 \\
0 & 1 & -2 & (2a - 3b)/2 \\
0 & 0 & 0 & (3b - 2a)/2
\end{pmatrix}.
\]
This system will be consistent if and only if \(3b - 2a = 0\).

4.32. The row echelon form is
\[
\begin{pmatrix}
1 & 3 & -2 \\
2 & 8 & x \\
0 & 8 & 5
\end{pmatrix} \sim \begin{pmatrix}
1 & 3 & -2 \\
0 & 8 & 5 \\
0 & 0 & x + 11/4
\end{pmatrix}.
\]
The matrix is invertible only if it is nonsingular. This is the case only if \(x + 11/4 \neq 0\), or equivalently, \(x \neq -11/4\).

4.33. The system
\[
\begin{align*}
2x + 3y &= 6 \\
4x + 6y &= 10
\end{align*}
\]
represents two parallel, distinct lines. There are no solutions.

4.34. The system
\[
\begin{align*}
x + 2y + 3x &= 6 \\
x + 2y + 3x &= 10
\end{align*}
\]
represents two parallel, distinct planes. The system has no solutions.
4.35. Suppose that $A$ is invertible. Then:

1. $A$ is nonsingular.
2. The only solution of the homogeneous system $Ay = 0$ is the zero vector $0$.
3. The equation $Ax = b$ has a unique solution for any right hand side $b$.
4. If $A$ is put into row echelon form then the diagonal entries of the result are nonzero.
5. If $A$ is put into reduced row echelon form then the result is the identity matrix.

4.36. Suppose that $A$ is nonsingular. Then:

1. $A$ is invertible.
2. The only solution of the homogeneous system $Ay = 0$ is the zero vector $0$.
3. The equation $Ax = b$ has a unique solution for any right hand side $b$.
4. If $A$ is put into row echelon form then the diagonal entries of the result are nonzero.
5. If $A$ is put into reduced row echelon form then the result is the identity matrix.

Section 5. Subspaces

5.1. The nullspace consists of solutions of

$$
\begin{pmatrix}
2 & -1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} = 0,
$$

or

$$
2x - y = 0.
$$

Solve for $x$.

$$
x = \frac{1}{2}y
$$

Hence, every element of the nullspace has the form

$$
x = \begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
(1/2)y \\
y
\end{pmatrix} = \begin{pmatrix}
y(1/2) \\
y
\end{pmatrix} ,
$$

where $y$ is free.

5.2. The nullspace consists of solutions of

$$
\begin{pmatrix}
-3 & 5
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} = 0,
$$

or

$$
-3x + 5y = 0.
$$

Solve for $x$

$$
x = \frac{5}{3}y
$$

Hence, every element of the nullspace has the form

$$
x = \begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
(5/3)y \\
y
\end{pmatrix} = \begin{pmatrix}
y(5/3) \\
y
\end{pmatrix} ,
$$

where $y$ is free.

5.3. The nullspace consists of solutions of

$$
\begin{pmatrix}
4 & 4 \\
-2 & -2
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
$$

Set up the augmented matrix and reduce.

$$
\begin{pmatrix}
4 & 4 & 0 \\
-2 & -2 & 0
\end{pmatrix} \sim \begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}$$
Hence, $y$ is free and

\[ x + y = 0 \]
\[ x = -y. \]

Therefore, all solutions have the form

\[ x = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \]

where $y$ is free.

5.4. The nullspace consists of solutions of

\[ \begin{pmatrix} 4 & 4 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

Set up the augmented matrix and reduce.

\[ \begin{pmatrix} 4 & 4 & 0 \\ -2 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]

Thus, $x = 0$ and $y = 0$ and the nullspace consists of a single zero vector.

5.5. We use row operations to reduce $A$ to row echelon form

\[ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

Hence $z$ is a free variable. We backsolve to find that $y = 0$ and $x = -y - z = -z$. Hence the nullspace consists of all vectors of the form

\[ v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}. \]

5.6. The nullspace consists of solutions of

\[ \begin{pmatrix} -3 & 8 & -11 \\ -4 & 10 & -14 \\ -2 & 5 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \]

We need only reduce the coefficient matrix if we mentally augment a zero vector in the fourth column.

\[ \begin{pmatrix} -3 & 8 & -11 \\ -4 & 10 & -14 \\ -2 & 5 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

Therefore, $x = y = z = 0$ and the nullspace consists of only the zero vector.

5.7. We use row operations to reduce $A$ to row echelon form

\[ \begin{pmatrix} 2 & -12 & -4 & -14 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]

We set the free variables $y_3 = s$ and $y_4 = t$. Then we backsolve to find that $y_2 = -y_4 = -t$, and then that $y_1 = [12y_2 + 4y_3 + 14y_4]/2 = 2s + t$. Hence the nullspace consists of all vectors of the form

\[ y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = s \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}. \]
5.8. The nullspace contains solutions of
\[
\begin{pmatrix}
-8 & 14 & -24 & 14 \\
4 & -10 & 18 & -10 \\
4 & -8 & 14 & -8 \\
-2 & 5 & -9 & 5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]
Reduce the matrix.
\[
\begin{pmatrix}
1 & 0 & -1/2 & 0 \\
0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
This gives
\[
x_1 - \frac{1}{2}x_3 = 0 \\
x_2 - 2x_3 + x_4 = 0
\]
Solve each equation for its pivot variable.
\[
x_1 = \frac{1}{2}x_3 \\
x_2 = 2x_3 - x_4
\]
Thus, solutions have the form
\[
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} (1/2)x_3 \\ 2x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1/2 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix},
\]
where \(x_3\) and \(x_4\) are free.

5.9. It is easy to see that \((1, 2)^T\) is not a scalar multiple of \((-1, 3)^T\). Thus, the vectors are independent. More formally, if
\[
c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]
then the augmented matrix reduces as follows.
\[
\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]
Thus, \(c_1 = c_2 = 0\) and the vectors are independent.

5.10. It is easy to see that \((-2, 3)^T\) is not a scalar multiple of \((2, -6)^T\). Thus, the vectors are independent. More formally, if
\[
c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]
then the augmented matrix reduces as follows.
\[
\begin{pmatrix} -2 & 2 & 0 \\ 3 & -6 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]
Thus, \(c_1 = c_2 = 0\) and the vectors are independent.

5.11. The vector \((-1, 7, 7)^T\) is not a scalar multiple of the vector \((-3, 7, -4)^T\). They are independent.

5.12. The vector \(v_1 = (-8, 9, -6)^T\) is not a scalar multiple of the vector \(v_2 = (-2, 0, 7)^T\). They are independent.

5.13. Set up the equation
\[
c_1 \begin{pmatrix} -1 \\ 7 \\ 7 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 7 \\ -4 \end{pmatrix} + c_3 \begin{pmatrix} -14 \\ 23 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]
The augmented matrix reduces as follows.
\[
\begin{pmatrix}
-1 & -3 & -4 & 0 \\
7 & 7 & -14 & 0 \\
7 & -4 & 23 & 0 \\
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]
Therefore, \(c_1 = c_2 = c_3 = 0\) and the vectors are independent.

5.14. Set up the equation
\[
c_1 \begin{pmatrix}
-8 \\
9 \\
-6 \\
\end{pmatrix} + c_2 \begin{pmatrix}
-2 \\
0 \\
7 \\
\end{pmatrix} + c_3 \begin{pmatrix}
8 \\
-18 \\
40 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}.
\]
The augmented matrix reduces as follows.
\[
\begin{pmatrix}
-8 & -2 & 8 & 0 \\
9 & 0 & -18 & 0 \\
-6 & 7 & 40 & 0 \\
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & -2 & 0 \\
0 & 1 & 4 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Thus,
\[
c_1 = 2c_3 \\
c_2 = -4c_3 \\
c_c = \text{free}
\]
For example, select \(c_3 = 1\). Then, \((c_1, c_2, c_3) = (2, -4, 1)\) and the calculation
\[
2 \begin{pmatrix}
-8 \\
9 \\
-6 \\
\end{pmatrix} - 4 \begin{pmatrix}
-2 \\
0 \\
7 \\
\end{pmatrix} + 1 \begin{pmatrix}
8 \\
-18 \\
40 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}
\]
shows that the vectors are dependent.

5.15. Set up the equation
\[
c_1 \begin{pmatrix}
-1 \\
7 \\
7 \\
\end{pmatrix} + c_2 \begin{pmatrix}
-3 \\
8 \\
-4 \\
\end{pmatrix} + c_3 \begin{pmatrix}
-4 \\
-14 \\
23 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}.
\]
Reduce the coefficient matrix.
\[
\begin{pmatrix}
-1 & -3 & -4 \\
7 & 8 & -14 \\
7 & -4 & 23 \\
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]
Thus, \(c_1 = c_2 = c_3 = 0\) and the vectors are independent.

5.16. Set up the equation
\[
c_1 \begin{pmatrix}
-8 \\
9 \\
-6 \\
\end{pmatrix} + c_2 \begin{pmatrix}
-2 \\
1 \\
7 \\
\end{pmatrix} + c_3 \begin{pmatrix}
8 \\
-18 \\
40 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}.
\]
Reduce the coefficient matrix.
\[
\begin{pmatrix}
-1 & -1 & 8 \\
9 & -1 & -18 \\
-6 & 7 & 40 \\
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]
Thus, \(c_1 = c_2 = c_3 = 0\) and the vectors are independent.

5.17. Since the vectors \(v_1\) and \(v_2\) are linearly independent, they form a basis for their span. Since there are two vectors in the basis, the dimension is 2.

5.18. Place the vectors in a matrix and reduce.
\[
\begin{pmatrix}
-2 & 2 \\
3 & 6 \\
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}
\]
Keep the columns that have nonzero pivots in the reduced matrix. Thus, a basis for \(\text{span}\{v_1, v_2\}\) is
\[
B = \left\{ \begin{pmatrix}
-2 \\
3 \\
\end{pmatrix}, \begin{pmatrix}
2 \\
6 \\
\end{pmatrix} \right\}.
\]
The dimension of the span is 2.

5.19. Place the vectors in a matrix and reduce.
\[
\begin{pmatrix}
-1 & -3 \\
7 & 7 \\
7 & -4
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}
\]
Keep the columns that have nonzero pivots in the reduced matrix. Thus, a basis for \(\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}\) is
\[
B = \left\{ \begin{pmatrix} -1 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} -3 \\ 7 \\ 7 \end{pmatrix} \right\}.
\]
The dimension of the span is 2.

5.20. Since \(\mathbf{v}_1, \mathbf{v}_2,\) and \(\mathbf{v}_3\) are linearly dependent, they do not form a basis. From Exercise 12 we know that 
\[
2\mathbf{v}_1 - 4\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0},
\] or \(\mathbf{v}_3 = 4\mathbf{v}_2 - 2\mathbf{v}_1\). Hence any vector in the span of \(\mathbf{v}_1, \mathbf{v}_2,\) and \(\mathbf{v}_3\) can be written as
\[
\begin{align*}
a_1\mathbf{v}_1 &+ a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3(4\mathbf{v}_2 - 2\mathbf{v}_1) \\
&= (a_1 - 2a_3)\mathbf{v}_1 + (a_2 + 4a_3)\mathbf{v}_2.
\end{align*}
\]
Thus every vector in the span of \(\mathbf{v}_1, \mathbf{v}_2,\) and \(\mathbf{v}_3\) is also in the span of \(\mathbf{v}_1\) and \(\mathbf{v}_2\). Furthermore, since the vectors \(\mathbf{v}_1\) and \(\mathbf{v}_2\) are not multiples of each other, they are linearly independent. Hence they form a basis of the span of \(\mathbf{v}_1, \mathbf{v}_2,\) and \(\mathbf{v}_3\). The dimension of the span is 2.

5.21. Place the vectors in a matrix and reduce.
\[
\begin{pmatrix}
-1 & -3 & -4 \\
7 & 7 & -14 \\
7 & -4 & 23
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
Thus, the vectors are independent and a basis for \(\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\) is
\[
B = \left\{ \begin{pmatrix} -1 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} -3 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} -4 \\ -14 \\ 23 \end{pmatrix} \right\}.
\]
The dimension is 3.

5.22. Place the vectors in a matrix and reduce.
\[
\begin{pmatrix}
-8 & -2 & 8 \\
9 & 0 & -18 \\
-6 & 7 & 40
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & -2 \\
0 & 1 & 4 \\
0 & 0 & 0
\end{pmatrix}
\]
Keep the vectors that have nonzero pivots in the reduced matrix. Thus, a basis for \(\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\) is
\[
B = \left\{ \begin{pmatrix} -8 \\ 9 \\ -6 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 7 \end{pmatrix} \right\}.
\]
The dimension is 2.

5.23. Place the vectors in a matrix and reduce.
\[
\begin{pmatrix}
-1 & -3 & -4 \\
7 & 8 & -14 \\
7 & -4 & 23
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
The vectors are independent and a basis for \(\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\) is
\[
B = \left\{ \begin{pmatrix} -1 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} -3 \\ 8 \\ -4 \end{pmatrix}, \begin{pmatrix} -4 \\ -14 \\ 23 \end{pmatrix} \right\}.
\]
The dimension is 3.

5.24. Place the vectors in a matrix and reduce.
\[
\begin{pmatrix}
-8 & -2 & 8 \\
9 & -1 & -18 \\
-6 & 7 & 40
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
The vectors are independent and a basis for span\{v_1, v_2, v_3\} is
\[ B = \left\{ \begin{pmatrix} -8 \\ 9 \\ -6 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 7 \end{pmatrix}, \begin{pmatrix} 8 \\ 18 \\ 40 \end{pmatrix} \right\}. \]

The dimension is 3.

\textbf{5.25.} The nullspace contains solutions of
\[ \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0. \]
Thus,
\[ 2x - y = 0, \]
\[ x = \frac{1}{2}y. \]
So solutions have the form
\[ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (1/2)y \\ y \end{pmatrix}, \]
where \( y \) is free. Thus a basis for the nullspace is
\[ B = \left\{ \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \right\}. \]

\textbf{5.26.} The nullspace contains solutions of
\[ \begin{pmatrix} -3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0. \]
Thus,
\[ -3x + 5y = 0, \]
\[ x = \frac{5}{3}y, \]
so solutions have the form
\[ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (5/3)y \\ y \end{pmatrix}, \]
where \( y \) is free. Thus, a basis for the nullspace is
\[ B = \left\{ \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} \right\}. \]

\textbf{5.27.} The nullspace contains solutions of
\[ \begin{pmatrix} 4 & 4 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]
The reduced form of the coefficient matrix is
\[ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}. \]
Thus,
\[ x + y = 0 \]
\[ x = -y, \]
and solutions have the form
\[ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \]
where \( y \) is free. Thus, a basis for the nullspace is
\[ B = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}. \]
5.28. The nullspace contains solutions of
\[
\begin{pmatrix} 4 & 4 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

The reduced form of the coefficient matrix is
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Thus, the nullspace contains only the zero vector and a basis is
\[ B = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}. \]

5.29. The nullspace contains solutions of
\[
\begin{pmatrix} 1 & 1 & 1 \\ -2 & -5 & -2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

The reduced form of the coefficient matrix is
\[
\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
Thus,
\[ x_1 = -x_3, \quad x_2 = 0 \]
Let \( x_3 = 1 \) to produce the basis.
\[ B = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \]

5.30. The null space contains solutions of
\[
\begin{pmatrix} -3 & 8 & -11 \\ -4 & 10 & -14 \\ -2 & 5 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

The reduced form of the coefficient matrix is
\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
Thus, the nullspace contains only the zero vector and a basis is
\[ B = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}. \]

5.31. In Exercise 7 we found that the nullspace consists of all vectors of the form
\[ y = s \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \]

Let \( v_1 = (2, 0, 1)^T \) and \( v_2 = (1, -1, 0)^T \). We see that \( v_1 \) and \( v_2 \) span the nullspace. In addition they are not multiples of each other, so they form a basis.
5.32. The nullspace contains solutions of
\[
\begin{pmatrix}
-8 & 14 & -24 & 14 \\
4 & 10 & 18 & -10 \\
4 & -8 & 14 & -8 \\
-2 & 5 & -9 & 5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]
The reduced form of the coefficient matrix is
\[
\begin{pmatrix}
1 & 0 & -1/2 & 0 \\
0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
Thus,
\[x_1 = \frac{1}{2}x_3,\]
\[x_2 = 2x_3 - x_4.\]
Letting \(x_3 = 1\) and \(x_4 = 0\), then \(x_3 = 0\) and \(x_4 = 1\), gives a basis
\[
B = \left\{ \begin{pmatrix} 1/2 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.
\]
5.33. By observation we find that \(v_p = (1, 0)^T\) is a particular solution to the system. In Exercise 1 we found that the nullspace is generated by \(v_h = (-1, 1)^T\). Hence the solution space to the system \(Ax = b\) consists of all vectors of the form \(x = v_p + tv_h\), where \(t \in \mathbb{R}\).

5.34. The equation
\[-3x + 5y = 2\]
has particular solution \(x_p = (-2/3, 0)^T\). We found a basis for the nullspace in Exercise 26.
\[
B = \left\{ \begin{pmatrix} 5/3 \\ 1 \end{pmatrix} \right\}
\]
Thus all solutions of \(-3x + 5y = 2\) are given by
\[
x = \begin{pmatrix} -2/3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 5/3 \\ 1 \end{pmatrix},
\]
where \(t \in \mathbb{R}\).

5.35. The reduced form of the augmented matrix \([A, b]\) is
\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}.
\]
Setting the free variable \(x_2 = 0\), we get the particular solution
\[
x_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
The reduced form of \(A\) is
\[
\begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix}.
\]
Setting the free variable \(x_2 = 1\), we get a basis for \(\text{null}(A)\).
\[
B = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}
\]
Thus, all solutions of \(Ax = b\) are given by
\[
x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix},
\]
\(t \in \mathbb{R}\).
5.36. The reduced form of the augmented matrix $[A, b]$ is
\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & -1
\end{pmatrix}.
\]
Thus, we have a unique solution $(1, -1)^T$.

5.37. The reduced form of the augmented matrix $[A, b]$ is
\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
Setting the free variable $x_3 = 0$ gives a particular solution
\[ x_p = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]
The reduced form of $A$ is
\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
Setting the free variable $x_3 = 1$, we get a basis for the nullspace.
\[ B = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.
\]
Thus, all solutions of $Ax = b$ are given by
\[ x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \]
$t \in \mathbb{R}$.

5.38. The reduced form of the augmented matrix $[A, b]$ is
\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
Setting the free variable $x_3 = 0$ gives a particular solution
\[ x_p = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.
\]
The reduced form of $A$ is
\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{pmatrix}.
\]
Setting the free variable $x_3 = 1$, we get a basis for the nullspace.
\[ B = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}.
\]
Thus, all solutions of $Ax = b$ are given by
\[ x = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \]
$t \in \mathbb{R}$.
5.39. By observation we find that $v_p = (3, 0, 0, 0)^T$ is a particular solution. In Exercise 31 we discovered that the nullspace of $A$ has as a basis the vectors $v_1 = (2, 0, 1, 0)^T$ and $v_2 = (1, -1, 0, 1)^T$. Hence every solution to the system $Ax = b$ is of the form $x = v_p + sv_1 + tv_2$, where $s, t \in \mathbb{R}$.

5.40. The reduced form of the augmented matrix $[A \ b]$ is

$$
\begin{pmatrix}
1 & 0 & -1/2 & 0 & -1/2 \\
0 & 1 & -2 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Setting the free variables $x_3 = x_4 = 0$, we get a particular solution

$$
x_p = \begin{pmatrix}
-1/2 \\
-1 \\
0 \\
0
\end{pmatrix}.
$$

The reduced form of $A$ is

$$
\begin{pmatrix}
1 & 0 & -1/2 & 0 \\
0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

Setting the free variables to $x_3 = 1$ and $x_4 = 0$, then $x_3 = 0$ and $x_4 = 1$, gives a basis for the nullspace.

$$
B = \left\{ \begin{pmatrix} 1/2 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\}
$$

Thus, all solutions of $Ax = b$ are given by

$$
x = \begin{pmatrix}
-1/2 \\
-1 \\
0 \\
0
\end{pmatrix} + s \begin{pmatrix}
1/2 \\
2 \\
0 \\
1
\end{pmatrix} + t \begin{pmatrix}
0 \\
-1 \\
0 \\
0
\end{pmatrix},
$$

$s, t \in \mathbb{R}$. 
Section 6. Determinants

6.1. The area of the triangle spanned by the vectors \( \mathbf{x}_1 = (x_1, y_1)^T \) and \( \mathbf{x}_2 = (x_2, y_2)^T \) is calculated by subtracting the areas of the triangular regions I, II, and III from the area of the bounding rectangle.

Thus, the area of the triangle is

\[
A = x_1y_2 - \frac{1}{2}x_2y_2 - \frac{1}{2}(x_1 - x_2)(y_2 - y_1) - \frac{1}{2}x_1y_1,
\]

\[
= \frac{1}{2}x_1y_2 - \frac{1}{2}x_2y_1,
\]

\[
= \frac{1}{2} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}.
\]

6.2. Estimate the area by counting square units inside the parallelogram in

The determinant is

\[
|v_1, v_2| = \begin{vmatrix} 9 & 1 \\ 1 & 9 \end{vmatrix} = (9)(9) - (1)(1) = 81 - 1 = 80.
\]

6.3. Estimate the area by counting square units inside the parallelogram in
The determinant is
\[ |v_1, v_2| = \begin{vmatrix} 1 & 6 \\ 4 & 1 \end{vmatrix} = (1)(1) - (4)(6) = 1 - 24 = -23. \]

Note that the determinant is the negative of the area.

6.4. Estimate the area by counting square units inside the parallelogram in

\[ (-2, 5), (4, 3) \]

The determinant is
\[ |v_1, v_2| = \begin{vmatrix} -2 & 4 \\ 5 & 3 \end{vmatrix} = (-2)(3) - (5)(4) = -6 - 20 = -26. \]

Note that the determinant is the negative of the area.

6.5. Estimate the area by counting square units inside the parallelogram in

\[ (-2, 6), (5, 5) \]
The determinant is
\[
\begin{vmatrix}
5 & -2 \\
5 & 6 \\
\end{vmatrix} = (5)(6) - (5)(-2) = 30 + 10 = 40.
\]

6.6. First note the determinant of \( A \).
\[
|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.
\]

To prove part (1), construct \( B \) by adding \( r \) times row 1 to row 2.
\[
B = \begin{pmatrix}
a + ra & b \\
c + ra & d + rb \\
\end{pmatrix}.
\]

Then,
\[
|B| = \begin{vmatrix} a & b \\ c + ra & d + rb \end{vmatrix},
\]
\[
= ad + rb - bc - ra,
\]
\[
= ad - bc,
\]
\[
= |A|.
\]

To prove part (2), craft \( B \) by swapping rows 1 and 2 of matrix \( A \).
\[
B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}
\]

Then,
\[
|B| = \begin{vmatrix} c & d \\ a & b \end{vmatrix},
\]
\[
= bc - ad,
\]
\[
= -(ad - bc),
\]
\[
= -|A|.
\]

Finally, to prove part (3), build matrix \( B \) by multiplying row 1 of matrix \( A \) by \( r \).
\[
B = \begin{pmatrix} ra & rb \\ c & d \end{pmatrix}
\]

Then,
\[
|B| = \begin{vmatrix} ra & rb \\ c & d \end{vmatrix},
\]
\[
= rad - rbc,
\]
\[
= r(ad - bc),
\]
\[
= r|A|.
\]

6.7. Using row operations we get
\[
\begin{pmatrix}
3 & 0 & 0 \\ 3 & 6 & 3 \\ -18 & -18 & -9
\end{pmatrix} \rightarrow \begin{pmatrix}
3 & 0 & 0 \\ 0 & 6 & 3 \\ -18 & -18 & -9
\end{pmatrix}
\]
\[
\rightarrow \begin{pmatrix}
3 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & -18 & -9
\end{pmatrix} \rightarrow \begin{pmatrix}
3 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 0 & 0
\end{pmatrix}
\]

Since the final matrix has a row of zeros, the determinant is 0.

6.8. Add \( 4/5 \) times row 1 to row 2; add \( -4/5 \) times row 1 to row 3.
\[
\begin{vmatrix}
5 & 6 & 4 \\ -4 & -9 & -8 \\ 4 & 6 & 5
\end{vmatrix} \rightarrow \begin{vmatrix}
5 & 6 & 4 \\ 0 & -21/5 & -24/5 \\ 0 & 6/5 & 9/5
\end{vmatrix}
\]
Add 2/7 times row 2 to row 3.

\[
\begin{bmatrix}
5 & 6 & 4 \\
0 & -21/5 & -24/5 \\
0 & 0 & 3/7 \\
\end{bmatrix}
= 5 \begin{bmatrix}
-21/5 \\
3/7 \\
\end{bmatrix}
= -9
\]

6.9. Add 3 times row 1 to row 2; add \(-4\) times row 1 to row 3.

\[
\begin{bmatrix}
1 & 0 & 4 \\
-3 & 3 & -2 \\
4 & -1 & -2 \\
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 4 \\
0 & 3 & 10 \\
0 & -1 & -18 \\
\end{bmatrix}
\]

Add 1/3 times row 2 to row 3.

\[
= \begin{bmatrix}
1 & 0 & 4 \\
0 & 3 & 10 \\
0 & 0 & -44/3 \\
\end{bmatrix}
= (1)(3) \begin{bmatrix}
-44/3 \\
\end{bmatrix}
= -44
\]

6.10. Using row operations we get

\[
\begin{bmatrix}
1 & 2 & -3 \\
0 & 6 & -2 \\
-2 & 3 & 2 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & -3 \\
0 & 6 & -2 \\
0 & 7 & -4 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & -3 \\
0 & 1 & -1 \\
0 & 0 & -5/3 \\
\end{bmatrix}
\]

The determinant of the final, upper triangular matrix is the product of the diagonal elements, or \(-10\).

6.11. Swap rows 1 and 3.

\[
\begin{bmatrix}
2 & -1 & 3 & 4 \\
0 & 2 & -2 & 0 \\
-1 & 2 & 0 & 0 \\
-1 & 3 & 1 & 2 \\
\end{bmatrix}
= \begin{bmatrix}
-1 & 2 & 0 & 0 \\
0 & 2 & -2 & 0 \\
2 & -1 & 3 & 4 \\
-1 & 3 & 1 & 2 \\
\end{bmatrix}
\]

Add 2 times row 1 to row 3; add \(-1\) times row 1 to row 4.

\[
= \begin{bmatrix}
-1 & 2 & 0 & 0 \\
0 & 2 & -2 & 0 \\
0 & 3 & 3 & 4 \\
0 & 1 & 1 & 2 \\
\end{bmatrix}
\]

Factor a 2 out of row 2.

\[
= -2 \begin{bmatrix}
-1 & 2 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 3 & 3 & 4 \\
0 & 1 & 1 & 2 \\
\end{bmatrix}
\]

Add \(-3\) times row 2 to row 3; add \(-1\) times row 2 to row 4.

\[
= -2 \begin{bmatrix}
-1 & 2 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 6 & 4 \\
0 & 0 & 2 & 2 \\
\end{bmatrix}
\]

Swap rows 3 and 4; then factor out a 2 from the new row 3.

\[
= (2)(2) \begin{bmatrix}
-1 & 2 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 6 & 4 \\
\end{bmatrix}
\]
Add −6 times row 3 to row 4.

\[
\begin{vmatrix}
-1 & 2 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & -2
\end{vmatrix} = (2)(2) \\
\begin{vmatrix}
-1 & 2 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & -2
\end{vmatrix} = (2)(2)(-1)(1)(1)(-2) \\
= 8
\]


\[
\begin{vmatrix}
3 & -3 & -2 & -1 \\
2 & 0 & -2 & -1 \\
1 & -2 & 0 & 0 \\
4 & -1 & -4 & -2
\end{vmatrix} \rightarrow \begin{vmatrix}
1 & -2 & 0 & 0 \\
2 & 0 & -2 & -1 \\
3 & -3 & -2 & -1 \\
4 & -1 & -4 & -2
\end{vmatrix}
\]

Add −2 times row 1 to row 2; add −3 times row 1 to row 3; add −4 times row 1 to row 4.

\[
\begin{vmatrix}
1 & -2 & 0 & 0 \\
0 & 4 & -2 & -1 \\
0 & 3 & -2 & -1 \\
0 & 7 & -4 & -2
\end{vmatrix} = -4
\]

Factor out a 4 form row 2.

\[
\begin{vmatrix}
1 & -2 & 0 & 0 \\
0 & 1 & -1/2 & -1/4 \\
0 & 3 & -2 & -1 \\
0 & 7 & -4 & -2
\end{vmatrix} = -4
\]

Add −3 times row 2 to row 3; add −7 times row 2 to row 4.

\[
\begin{vmatrix}
1 & -2 & 0 & 0 \\
0 & 1 & -1/2 & -1/4 \\
0 & 0 & -1/2 & -1/4 \\
0 & 0 & -1/2 & -1/4
\end{vmatrix} = -4
\]

Add −1 times row 3 to row 4.

\[
\begin{vmatrix}
1 & -2 & 0 & 0 \\
0 & 1 & -1/2 & -1/4 \\
0 & 0 & -1/2 & -1/4 \\
0 & 0 & 0 & 0
\end{vmatrix} = -4(1)(1)\left(-\frac{1}{2}\right) \left(0\right) \\
= 0
\]

6.13. (a) Suppose that row \(i\) is a scalar multiple of row \(j\), say

\[
|A| = \begin{vmatrix}
\vdots & \vdots & \vdots \\
ra_{j1} & ra_{j2} & \ldots & ra_{jn} \\
\vdots & \vdots & \vdots \\
a_{j1} & a_{j2} & \ldots & a_{jn} \\
\vdots & \vdots & \vdots
\end{vmatrix}
\]

\[
= ra_{j1} \begin{vmatrix}
\vdots & \vdots & \vdots \\
a_{j1} & a_{j2} & \ldots & a_{jn} \\
\vdots & \vdots & \vdots
\end{vmatrix}
\]

\[
= 0
\]
Then, adding \(-r\) times for \(j\) to row \(i\) does not change the value of the determinant. Thus,

\[
\begin{vmatrix}
\vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots \\
a_{i1} & a_{i2} & \ldots & a_{in} \\
\vdots & \vdots & \vdots \\
\end{vmatrix}
= 0,
\]

because it has a zero row. The same result applies to the columns. Suppose column \(i\) is a scalar multiple of column \(j\), say

\[
|A| = \begin{vmatrix}
\vdots & r a_{1j} & \ldots & a_{1j} & \ldots \\
\vdots & r a_{2j} & \ldots & a_{2j} & \ldots \\
\vdots & \vdots & \vdots \\
\vdots & r a_{nj} & \ldots & a_{nj} & \ldots \\
\end{vmatrix}
\]

But adding \(-r\) times column \(j\) to column \(i\) does not change the value of the determinant. Thus,

\[
\begin{vmatrix}
\vdots & 0 & \ldots & a_{1j} & \ldots \\
\vdots & 0 & \ldots & a_{2j} & \ldots \\
\vdots & \vdots & \vdots \\
\vdots & 0 & \ldots & a_{nj} & \ldots \\
\end{vmatrix}
= 0,
\]

because \(A\) has a zero column.

(b) The first matrix has zero determinant because the third row has all zeros. Similarly, the second matrix has zero determinant because the third column has all zeros. The third matrix has zero determinant because its second row is a scalar multiple of its first row. The fourth matrix has zero determinant because the second column is a scalar multiple of its first column.

6.14. (a) Suppose matrix \(A = [a_1, a_2, \ldots, a_n]\) is \(n \times n\), where \(a_j\) denotes the \(j\)th column. Assume that column \(j\) is a linear combination of the columns that proceed it,

\[a_j = c_1 a_1 + \cdots + c_{j-1} a_{j-1}.\]

Thus,

\[a_j - c_1 a_1 - \cdots - c_{j-1} a_{j-1} = 0.\]

Thus, if we perform the following

1. Add \(-c_1\) times column 1 and add to column \(j\).
2. Add \(-c_2\) times column 1 and add to column \(j\).

\[\vdots\]

\((j - 1)\) Add \(-c_{j-1}\) times column 1 to column \(j\).

Then,

\[
|a_1, a_2, \ldots, a_j, \ldots, a_n|,
= |a_1, a_2, \ldots, (a_j - c_1 a_1 - \cdots - c_{j-1} a_{j-1}), \ldots, a_n|,
= |a_1, a_2, \ldots, 0, \ldots, a_n|,
= 0,
\]

because there is a zero column.
If the \( j \)th row of \( A \) is a linear combination of its preceding rows, then the \( j \)th column of \( A^T \) is a linear combination of its preceding columns and has zero determinant. Thus,

\[
|A| = |A^T| = 0.
\]

(b) The first matrix has zero determinant because its third row is the sum of its first two rows. The second matrix has zero determinant because its third column is the sum of its first two columns. Matrix three has zero determinant because its third row equals twice its first row added to its second row. The fourth matrix has zero determinant because its third column is the sum of twice its first column and three times its second column.

6.15. The transpose of a lower triangular matrix is upper triangular. The determinant of the transpose equals the determinant of the original. Note that transposing a lower triangular matrix does not dislodge the diagonal elements. Thus, the determinant of a lower triangular matrix is the product of its diagonal elements.

\[
|A| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 4 \end{vmatrix} = (1)(2)(3)(4) = 24
\]

6.16. Since there are two zeros in the first row we expand by that row.

\[
\det A = 3 \det \begin{pmatrix} 6 & 3 \\ -18 & -9 \end{pmatrix} = 3(-54 + 54) = 0.
\]

6.17. Expand across the first row.

\[
\begin{vmatrix} 5 & 6 & 4 \\ -4 & -9 & -8 \\ 4 & 6 & 5 \end{vmatrix} = 5 \begin{vmatrix} -9 & -8 \\ 6 & 5 \end{vmatrix} - 6 \begin{vmatrix} -4 & -8 \\ 4 & 5 \end{vmatrix} + 4 \begin{vmatrix} -4 & -9 \\ 4 & 6 \end{vmatrix} \\
= 5(-45 + 48) - 6(-20 + 32) + 4(-24 + 36) \\
= 15 - 72 + 48 \\
= -9
\]

6.18. Expand across the first row.

\[
\begin{vmatrix} 1 & 0 & 4 \\ -3 & 3 & -2 \\ 4 & -1 & -2 \end{vmatrix} = 1 \begin{vmatrix} 3 & -2 \\ 4 & -1 \end{vmatrix} + 4 \begin{vmatrix} -3 & 3 \\ 4 & -1 \end{vmatrix} \\
= 1(-6 - 2) + 4(3 - 12) \\
= -8 - 36 \\
= -44
\]

6.19. There is a zero in the first column, so we will expand by that column.

\[
\det A = 1 \cdot \det \begin{pmatrix} 6 & -2 \\ 3 & 2 \end{pmatrix} + (-2) \cdot \det \begin{pmatrix} 2 & -3 \\ 6 & -2 \end{pmatrix} \\
= 18 - 28 = -10
\]

6.20. Expand across the third row.

\[
\begin{vmatrix} 2 & -1 & 3 & 4 \\ 0 & 2 & -2 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 3 & 1 & 2 \end{vmatrix} = -1 \begin{vmatrix} -1 & 3 & 4 \\ 2 & 0 & 0 \\ 3 & 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 & 4 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{vmatrix} \\
= -1(-1 - 6) - 2(-2 + 0) \\
= -7 + 4 \\
= -3
\]
But, expanding across the second row of the first and second matrices,

\[
= -1 \left\{ -2 \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} - 1 \begin{vmatrix} -1 & 4 \\ 3 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} \right\} = 1(-2(2) - 2(-14)) - 2(-2(8)) = 4 - 28 + 32 = 8
\]

6.21. Expand across the third row.

\[
\begin{vmatrix} 3 & -3 & -2 & 1 \\ 2 & 0 & -2 & -1 \\ 1 & -2 & 0 & 0 \\ 4 & -1 & -4 & -2 \end{vmatrix} = 1 \begin{vmatrix} -3 & -2 & -1 \\ 0 & -2 & -1 \\ -1 & -4 & -2 \end{vmatrix} + 2 \begin{vmatrix} 3 & -2 \\ 2 & 4 \\ 4 & 4 \end{vmatrix}
\]

Expand down the first column of the first matrix and the first row of the second matrix.

\[
= 1 \left\{ -3 \begin{vmatrix} -2 & -1 \\ -4 & -2 \end{vmatrix} - 1 \begin{vmatrix} -2 & -1 \\ -4 & -2 \end{vmatrix} \right\} + 2 \left\{ 3 \begin{vmatrix} -2 & -1 \\ -4 & -2 \end{vmatrix} + 2 \begin{vmatrix} -2 & -1 \\ 4 & 4 \end{vmatrix} \right\}
\]

Note that in each \(2 \times 2\) determinant, the second row is a multiple of the first. Therefore, they all equal zero and the determinant of the original \(4 \times 4\) is also zero.

6.22. The determinant of \(A\) is

\[
\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 0 & 7 \end{vmatrix} = 7.
\]

To find the nullspace, reduce the augmented matrix \([A \: 0]\).

\[
\begin{pmatrix} 1 & -2 & 0 \\ 2 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

The nullspace contains only the zero vector. Thus, the only solution of

\[
x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

is \(x_1 = x_2 = 0\) and the columns are independent.

6.23. The determinant of \(A\) is

\[
|A| = \begin{vmatrix} -1 & 2 \\ 2 & -4 \end{vmatrix} = 0.
\]

To find the nullspace, reduce the augmented matrix \([A \: 0]\).

\[
\begin{pmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

The solutions are

\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2x_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix},
\]

where \(x_2\) is free. A basis for the nullspace is

\[
B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.
\]

Because the nullspace is nontrivial,

\[
x_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

has nontrivial solutions, so the columns of \(A\) are dependent.

6.24. The determinant of \(A\) is

\[
|A| = \begin{vmatrix} 1 & 3 \\ -1 & -3 \end{vmatrix} = 0.
\]
To find the nullspace, reduce the augmented matrix \([A \, \mathbf{0}]\).
\[
\begin{pmatrix}
1 & 3 & 0 \\
-1 & -3 & 0 \\
\end{pmatrix} \sim 
\begin{pmatrix}
1 & 3 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]
The solutions are
\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix},
\]
where \(x_2\) is free. Thus,
\[
B = \left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}
\]
is a basis for the nullspace. Because the nullspace is nontrivial,
\[
x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
has nontrivial solutions, so the columns of \(A\) are dependent.

6.25. The determinant of \(A\) is
\[
|A| = \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = 1
\]
To find the nullspace, reduce the augmented matrix \([A \, \mathbf{0}]\).
\[
\begin{pmatrix}
2 & -1 & 0 \\
1 & 0 & 0 \\
\end{pmatrix} \sim 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{pmatrix}
\]
The nullspace contains only the zero vector. Thus, the only solution of
\[
x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
is \(x_1 = x_2 = 0\) and the columns are independent.

6.26. Taking advantage of the zeros we expand by the third row.
\[
\det A = \det \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = -1.
\]
Since \(\det A \neq 0\), null\((A)\) is trivial and the column vectors of \(A\) are linearly independent.

6.27. The determinant is 0. The reduced row echelon form is
\[
\begin{pmatrix}
1 & -2 & -4 \\
2 & 1 & 2 \\
3 & 0 & 0 \\
\end{pmatrix} \sim 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
\end{pmatrix}
\]
A basis for the nullspace is
\[
B = \left\{ \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}.
\]
Thus, the equation
\[
x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]
has nontrivial solutions and the columns are dependent.

6.28. To take advantage of the zero we could expand by the first row or the second column. Let’s choose the second column.
\[
\det A = -1 \cdot \det \begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} = 1 - 1 = 0.
\]
Since \(\det A = 0\), null\((A)\) is nontrivial, and the column vectors of \(A\) are linearly dependent. null\((A)\) is the subspace with basis \((1, 1, -1)^T\).

6.29. The determinant is 1. The reduced row echelon form is
\[
\begin{pmatrix}
1 & 1 & 2 \\
-1 & 1 & 5 \\
1 & 0 & -1 \\
\end{pmatrix} \sim 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]
The nullspace contains only the zero vector. Thus, the equation
\[ x_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]
has only the trivial solution and the columns are independent.

6.30. To have a nontrivial nullspace, the determinant must equal zero.

\[
0 = \begin{vmatrix} 2 & x \\ 3 & -2 \end{vmatrix} \\
0 = -4 - 3x \\
x = -\frac{4}{3}
\]

6.31. To have a nontrivial nullspace, the determinant must equal zero.

\[
0 = \begin{vmatrix} 2 & x \\ x & 3 \end{vmatrix} \\
0 = 6 - x^2 \\
x = \pm\sqrt{6}
\]

6.32. To have a nontrivial nullspace the determinant must equal zero.

\[
0 = \begin{vmatrix} x & 4 \\ 3 & -2 \end{vmatrix} \\
0 = -2x - 12 \\
2x = -12 \\
x = -6
\]

6.33. To have a nontrivial nullspace, the determinant must equal zero.

\[
0 = \begin{vmatrix} -1 & x \\ -x & 4 \end{vmatrix} \\
0 = -4 + x^2 \\
x^2 = 4 \\
x = \pm 2
\]

6.34. To have a nontrivial nullspace, the determinant must equal zero.

\[
0 = \begin{vmatrix} 2 - x & 1 \\ 0 & -1 - x \end{vmatrix} \\
0 = (2 - x)(-1 - x) \\
x = 2, 1
\]

6.35. To have a nontrivial nullspace, the determinant must equal zero.

\[
0 = \begin{vmatrix} -1 - x & 0 \\ 3 & 2 - x \end{vmatrix} \\
0 = (-1 - x)(2 - x) \\
x = -1, 2
\]

6.36. To have a nontrivial nullspace, the determinant must equal zero.

\[
0 = \begin{vmatrix} -1 - x & 5 & 2 \\ 0 & -x & -1 \\ 0 & 6 & -5 - x \end{vmatrix}
\]
7.6. Determinants

Expand down the first column.

\[ 0 = (-1 - x) \begin{vmatrix} -x & -1 \\ 6 & -5 - x \end{vmatrix} \]
\[ = (-1 - x)(-x(-5 - x) + 6) \]
\[ = -(x + 1)(x^2 + 5x + 6) \]
\[ = -(x + 1)(x + 3)(x + 2) \]

Thus, \( x = -1, -3, -2. \)

6.37. To have a nontrivial nullspace, the determinant must equal zero.

\[ 0 = \begin{vmatrix} 2 - x & 0 & 0 \\ -1 & -x & 2 \\ 0 & -2 & 5 - x \end{vmatrix} \]

Expand across the first row.

\[ 0 = (2 - x) \begin{vmatrix} -x & 2 \\ -2 & 5 - x \end{vmatrix} \]
\[ = (2 - x)(-x(5 - x) + 4) \]
\[ = -(x - 2)(x^2 - 5x + 4) \]
\[ = -(x - 2)(x - 1)(x - 4) \]

Thus, \( x = 2, 1, 4. \)

6.38. To have a nontrivial nullspace, the determinant must equal zero.

\[ 0 = \begin{vmatrix} 2 - x & 0 & 1 \\ -3 & -1 - x & -1 \\ -2 & 0 & -1 - x \end{vmatrix} \]

Expand across the first row.

\[ 0 = (2 - x) \begin{vmatrix} -1 - x & -1 \\ 0 & -1 - x \end{vmatrix} + 1 \begin{vmatrix} -3 & -1 - x \\ -2 & 0 \end{vmatrix} \]
\[ = (2 - x)(-1 - x)(-1 - x) + 2(-1 - x) \]
\[ = (-1 - x)[(2 - x)(-1 - x) + 2] \]
\[ = (-1 - x)(x^2 - x) \]
\[ = -(x + 1)x(x - 1) \]

Thus, \( x = -1, 0, 1. \)

6.39. To have a nontrivial nullspace, the determinant must equal zero.

\[ 0 = \begin{vmatrix} -1 - x & 2 & 2 \\ 0 & -2 - x & 0 \\ -1 & 4 & 2 - x \end{vmatrix} \]

Expand across the second row.

\[ 0 = (-2 - x) \begin{vmatrix} -1 - x & 2 \\ -1 & 2 - x \end{vmatrix} \]
\[ = -(x + 2)((-1 - x)(2 - x) + 2) \]
\[ = -(x + 2)(-x + x^2) \]
\[ = -(x + 2)x(x - 1) \]

Thus, \( x = -2, 0, 1. \)

6.40. The determinant is

\[ \begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} = 3. \]

Therefore, the nullspace is trivial, containing only the zero vector.
6.41. The determinant is
\[
\begin{vmatrix}
-2 & 3 \\
-2 & 4
\end{vmatrix} = -2.
\]
Therefore, the nullspace is trivial, containing only the zero vector.

6.42. The determinant is
\[
\begin{vmatrix}
-1 & -9 & 10 \\
-7 & -19 & 26 \\
-2 & -10 & 12
\end{vmatrix} = 0.
\]
This is easily seen by noticing that the third column is the negative of the sum of the first two columns. The nullspace is not trivial. It contains nonzero vectors.

6.43. Add 3 times the first row to the second; add \(-4\) times the first row to the third.
\[
\begin{vmatrix}
1 & 0 & 4 \\
-3 & 3 & -2 \\
4 & 0 & -2
\end{vmatrix}
= \begin{vmatrix}
1 & 0 & 4 \\
0 & 3 & 10 \\
0 & 0 & -18
\end{vmatrix}
= -54
\]
The nullspace is trivial, containing only the zero vector.

6.44. Add 2 times row 2 to row 2; add \(-3\) times row 1 to row 3.
\[
\begin{vmatrix}
1 & 2 & -3 \\
-2 & 0 & -1 \\
3 & 0 & 2
\end{vmatrix}
= \begin{vmatrix}
1 & 2 & -3 \\
0 & 4 & -7 \\
0 & -6 & 11
\end{vmatrix}
\]
Add \(3/2\) times row 2 to row 3.
\[
= \begin{vmatrix}
1 & 2 & -3 \\
0 & 4 & -7 \\
0 & 0 & 1/2
\end{vmatrix}
= (1)(4) \left( \frac{1}{2} \right)
= 2
\]
The nullspace is trivial, containing only the zero vector.

6.45. Swap rows 1 and 3.
\[
\begin{vmatrix}
0 & 1 & 2 \\
2 & 0 & -2 \\
-1 & 0 & 3
\end{vmatrix}
= \begin{vmatrix}
-1 & 0 & 3 \\
2 & 0 & -2 \\
0 & 1 & 2
\end{vmatrix}
\]
Add 2 times row 1 to row 2.
\[
= \begin{vmatrix}
-1 & 0 & 3 \\
0 & 0 & 4 \\
0 & 1 & 2
\end{vmatrix}
\]
Swap rows 2 and 3.
\[
= \begin{vmatrix}
-1 & 0 & 3 \\
0 & 1 & 2 \\
0 & 0 & 4
\end{vmatrix}
= (-1)(1)(4)
= -4
\]
The nullspace is trivial, containing only the zero vector.

6.46. Swap columns 1 and 2.
\[
\begin{vmatrix}
10 & -1 & -3 & 9 \\
3 & 2 & -3 & 3 \\
3 & 1 & -2 & 3 \\
-10 & 2 & 2 & -9
\end{vmatrix}
= \begin{vmatrix}
-1 & 10 & -3 & 9 \\
2 & 3 & -3 & 3 \\
1 & 3 & -2 & 3 \\
2 & -10 & 2 & -9
\end{vmatrix}
\]
Add 2 times row 1 to row 2; add row 1 to row 3; add 2 times row 1 to row 4.

\[
\begin{vmatrix}
-1 & 10 & -3 & 9 \\
0 & 23 & -9 & 21 \\
0 & 13 & -5 & 11 \\
0 & 10 & -4 & 9 \\
\end{vmatrix} = - \begin{vmatrix}
23 & -9 & 21 \\
13 & -5 & 11 \\
10 & -4 & 9 \\
\end{vmatrix}
\]

Too hard to continue with row reductions. Expand down first column.

\[
\begin{vmatrix}
23 & -9 & 21 \\
13 & -5 & 11 \\
10 & -4 & 9 \\
\end{vmatrix} = (-1) \begin{vmatrix}
13 & 11 \\
10 & 9 \\
\end{vmatrix}
\]

Expand across first row.

\[
= 1 \begin{vmatrix}
23 & -5 & 11 \\
-4 & 9 \\
\end{vmatrix} + 9 \begin{vmatrix}
13 & 11 \\
10 & 9 \\
\end{vmatrix} + 21 \begin{vmatrix}
13 & -5 \\
10 & -4 \\
\end{vmatrix}
\]

\[
= 23(-1) + 9(7) - 21(-2)
\]

\[
= -23 + 63 - 42
\]

\[
= -2
\]

The nullspace is trivial and contains only the zero vector.

**6.47.** Swap rows 1 and 2.

\[
\begin{vmatrix}
3 & 0 & 20 & -8 \\
2 & 3 & -2 & 0 \\
6 & 4 & 17 & -8 \\
16 & 10 & 50 & -23 \\
\end{vmatrix} = - \begin{vmatrix}
2 & 3 & -2 & 0 \\
3 & 0 & 20 & -8 \\
6 & 4 & 17 & -8 \\
16 & 10 & 50 & -23 \\
\end{vmatrix}
\]

Add \(-3/2\) times row 1 to row 2; add \(-3\) times row 1 to row 3; add \(-8\) times row 1 to row 4.

\[
\begin{vmatrix}
2 & 3 & -2 & 0 \\
0 & -9/2 & 23 & -8 \\
0 & -5 & 23 & -8 \\
0 & -14 & 66 & -23 \\
\end{vmatrix} = - \begin{vmatrix}
2 & 3 & -2 & 0 \\
0 & -9/2 & 23 & -8 \\
0 & -5 & 23 & -8 \\
0 & -14 & 66 & -23 \\
\end{vmatrix}
\]

Too hard to continue with reduction. Expand down first column.

\[
\begin{vmatrix}
2 & 3 & -2 & 0 \\
0 & -9/2 & 23 & -8 \\
0 & -5 & 23 & -8 \\
0 & -14 & 66 & -23 \\
\end{vmatrix} = -2 \begin{vmatrix}
-9/2 & 23 & -8 \\
-5 & 23 & -8 \\
-14 & 66 & -23 \\
\end{vmatrix}
\]

Expand across first row.

\[
= 9 \begin{vmatrix}
23 & -8 \\
-9/2 & -23 \\
\end{vmatrix} + 46 \begin{vmatrix}
-5 & -8 \\
-14 & -23 \\
\end{vmatrix} + 16 \begin{vmatrix}
-5 & 23 \\
-14 & 66 \\
\end{vmatrix}
\]

\[
= 9(-1) + 46(3) + 16(-8)
\]

\[
= 1
\]

The nullspace is trivial, containing only the zero vector.

**6.48.** The second and fourth row are identical. The determinant is zero and the nullspace contains nonzero vectors.

**6.49.** Rows 2 and 5 are identical, so the determinant is 0. Therefore there is a nonzero vector in the nullspace.

**6.50.** (a) \(-2U\) multiplies each of 4 rows by \(-2\). Thus,

\[
\det(-2U) = (-2)^4 \det(U)
\]

\[
= (-2)^4(-3)
\]

\[
= -48
\]
(b) The determinant of a product is the product of the determinants. Thus, 
\[
\det(U^3) = \det(UUU) = \det(U) \det(U) \det(U) \\
= (-3)(-3)(-3) \\
= -27
\]

(c) Because 
\[
1 = \det(I) = \det(UU^{-1}) = \det(U) \det(U^{-1}),
\]
we know 
\[
\det(U^{-1}) = \frac{1}{\det U} = \frac{1}{-3} = -\frac{1}{3}.
\]

6.51. False. For a counterexample, let 
\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.
\]
Then 
\[
\det(A + B) = \det \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} = -2.
\]
But, 
\[
\det(A) + \det(B) = \det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = -1 + 0 = -1.
\]

6.52. (a) If \( A \) and \( B \) are similar, then there is a nonsingular \( S \) such that \( A = S^{-1}BS \). Thus, 
\[
\det A = \det(S^{-1}BS) \\
= \det(S^{-1}) \det(B) \det(S) \\
= \frac{1}{\det(S)} \det(B) \det(S) \\
= \det(B).
\]
(b) Then 
\[
\det(A - \lambda I) = \det(S^{-1}BS - \lambda I).
\]
But, \( I \) commutes with \( S \), so 
\[
= \det(S^{-1}BS - S^{-1}(\lambda I)S), \\
= \det(S^{-1}(B - \lambda I)S), \\
= \det(S^{-1}) \det(B - \lambda I) \det(S), \\
= \frac{1}{\det(S)} \det(B - \lambda I) \det(S), \\
= \det(B - \lambda I).
\]

6.53. \( \det A \neq 0 \) is equivalent to each of the following:

(1) \( A \) is nonsingular.
(2) \( A \) is invertible.
(3) \( \text{null}(A) \) is trivial.
(4) The system \( Ax = b \) has a unique solution for every right hand side \( b \).
(5) If \( A \) is \( n \times n \), the column vectors in \( A \) are a basis for \( \mathbb{R}^n \).
(6) When \( A \) is reduced to row echelon form the diagonal entries are all nonzero.
(7) When \( A \) is reduced to reduced row echelon form the result is the identity matrix.

6.54. \( \det A = 0 \) is equivalent to each of the following:

(1) \( A \) is singular.
(2) \( A \) is noninvertible.
(3) \( \text{null}(A) \) contains a nonzero vector.
(4) There is a vector \( \mathbf{b} \) for which the system \( A\mathbf{x} = \mathbf{b} \) is inconsistent.
(5) If \( A \) is \( n \times n \), the column vectors in \( A \) are linearly dependent.
(6) When \( A \) is reduced to row echelon form there is a row of zeros.