Exercises 5/29/08

1. Prove properties 7 and 8 using properties 1-6.

2. How many times do the two given curves intersect at the origin?

   (a) \( y = x^3 \) and \( y^4 + 6x^3y + x^8 = 0 \)
   (b) \( y = x^2 - 2x \) and \( y^2 + 5y = 4x^3 \)
   (c) \( y^2 + x^2y - x^3 = 0 \) and \( y^2 + x^3y + 2x = 0 \)
   (d) \( y^5 = x^7 \) and \( y^2 = x^3 \)
   (e) \( xy^4 + y^3 = x^2 \) and \( y^5 + x^2 = xy \)

3. Let \( C \) and \( D \) be two circles through the origin, and assume that the center of \( C \) lies on the \( x \)-axis. Prove that \( C \) and \( D \) intersect either once or twice at the origin, depending on whether or not the center of \( D \) lives on the \( x \)-axis.

4. Prove that if \( \deg(f) = m \) and \( \deg(g) = n \), then \( I_0(f, g) \leq mn \).

5. Show that the graph of the equation \( r = \sin(3\theta) \) in polar coordinates corresponds to a curve \( f(x, y) = 0 \) of degree 4. Prove that there exist 3 lines through the origin that intersect the curve more than 3 times there and that all other lines through the origin intersect the curve exactly 3 times there. Draw the curve and the 3 exceptional lines.

6. Prove that a plane curve of degree \( n \) with a point of multiplicity \( n - 1 \) is rational.

7. Let \( C \) be a curve and \( P \) a point on the curve. Then \( P \) is called a flex or inflection point of \( C \) if there exists a line \( L \) through \( P \) with intersection multiplicity \( \geq 3 \). Find a necessary and sufficient condition for there to exist a line \( L \) and a point \( P \) so that \( P \) is a flex point with respect to this line on the curve \( C \).

8. Show that affine equivalence preserves rationality. That is, show that if \( f(x, y) = g(\phi(x, y)) \) for some affine equivalence \( \phi \) and \( g(x, y) = 0 \) is rational then \( f(x, y) = 0 \) is also rational.

9. (a) Curves of degree 2 are called conics. Show that any conic \( f(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = 0 \) is affine equivalent to one of the form \( g(x, y) = ax + by + cx^2 + dxy + ey^2 \).
   (b) Using \#6 and \#8 conclude that any conic is rational.