§3.3 #10. Solution. \( \partial f/\partial x = \sin y \) and \( \partial f/\partial y = 1 + x \cos y \). The first is zero when \( y = \pi k \) where \( k \) is any integer. Now \( \cos(\pi k) \) equals 1 if \( k \) is even and \(-1\) if \( k \) is odd. So we get an infinite number of critical points, namely \((x, y) = (-1, \pi k)\) if \( k \) is even and \((x, y) = (1, \pi k)\) when \( k \) is odd. Now \( \partial f/\partial x = \sin y \) and \( \partial f/\partial y = 1 + \cos y \) while \( \partial^2 f/\partial y^2 = -x \sin y \). So the matrix of second partials is

\[
\begin{pmatrix}
0 & \cos y \\
\cos y & -x \sin y
\end{pmatrix}
\]

The determinant is \(-\cos^2 y\) so when evaluated at any of the critical points this is negative. Therefore all the critical points are saddle points.

§3.3 #22 Solution. A point in the plane \( 2x - y + 2z = 20 \) looks like \((x, 2x + 2z - 20, z)\) where we substituted for \( y \). we want to minimize \( \sqrt{x^2 + y^2 + z^2} \). This is the same as minimizing \( x^2 + y^2 + z^2 \) which should be computationally easier to do. Substituting for \( y \) we get the function

\[
g(x, z) = x^2 + (2x + 2z - 20)^2 + z^2
\]

where \( x, z \) can be any real numbers. Let’s find the critical points. \( \partial g/\partial x = 2x + 2(2x + 2z - 20)2 = 10x + 8z - 80 \) while \( \partial g/\partial z = 2(2x + 2z - 20)2 + 2z = 8x + 10z - 80 \). Solving

\[
10x + 8z - 80 = 8x + 10z - 80 = 0
\]

we get \( x = 5, y = 5 \) as the only critical point. Now the matrix of mixed partials can easily be computed to equal

\[
\begin{pmatrix}
10 & 8 \\
8 & 10
\end{pmatrix}
\]

which is positive definite (i.e., \( 10 > 0 \) and \( 100 - 64 > 0 \)). So \((5, 5)\) is a minimum for \( g(x, z) \) and hence the closest point to the origin is \((x, y, z) = (5, 0, 5)\).

§3.3 #34. Maximize \( f(x, y) = xy \) on the rectangle \([-1, 1] \times [-1, 1] \).
Solution. We first check for critical points in the interior of the rectangle. \( \nabla f = (y, x) \). This is \((0, 0)\) only if \( x = y = 0 \). So this is a possible extrema of the function \( f \). But we have yet to check the boundary. We can parametrize the boundary with the four curves

\[
c_1(t) = (t, -1) \quad c_2(t) = (1, t) \\
c_3(t) = (t, 1) \quad c_4(t) = (-1, t)
\]

where in each curve \(-1 \leq t \leq 1\). We now look for the min and max on the boundary by checking for the min and max of \( f(c_i(t)) \) for each \( i = 1, 2, 3, 4 \). For example \( f(c_1(t)) = -t \). This has minimum at \( t = 1 \) and max at \( t = -1 \). In other words the min occurs at \((1, -1)\) and the max occurs at \((-1, -1)\). Checking all four curves we see that the absolute max of \( f \) on the square occurs at \((1, 1)\) and \((-1, -1)\) and the absolute min occurs at \((1, -1)\) and \((-1, 1)\).