§8.2 #7. Let $\mathbf{F} = r \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$ where $r = xi + yj + zk$ is the position vector. Calculate $\iint_S \nabla \times \mathbf{F}.d\mathbf{S}$, where $S$ is the portion of the surface of a sphere defined by $x^2 + y^2 + z^2 = 1$ and $x + y + z \geq 1$.

**Solution.** By Stokes theorem, $\iint_S \nabla \times \mathbf{F}.d\mathbf{S} = \int_{\partial S} \mathbf{F}.d\mathbf{s}$. Let $\mathbf{c}(t)$ parametrize the curve $\partial S$ for $t \in [a,b]$. Then,

$$\int_{\partial S} \mathbf{F}.d\mathbf{s} = \int_a^b \mathbf{F}.\frac{\mathbf{c}'(t)}{||\mathbf{c}'(t)||}||\mathbf{c}'(t)||dt.$$ 

Since we don’t actually want to parametrize the curve, we will use geometry to study this integral. The curve $\partial S$ is given by the intersection of the plane and the sphere pictured below.

[Diagram of the intersection of a plane and a sphere]

The vector field $\mathbf{F}$ is given by $\mathbf{F} = r \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$. For every point $(x,y,z)$ on $\partial S$, $\mathbf{F}(x,y,z)$ is a vector tangent to $\partial S$ of length equal to the area of the parallelogram spanned by $(x,y,z)$ and $(1,1,1)$ pointing in the “clockwise” direction around $\partial S$. For every $(x,y,z)$ on $\partial S$, the length of this vector is constant. Using the point $(x,y,z) = (1,0,0)$, you can easily see that this value is $\sqrt{2}$. Therefore, using the formula $\mathbf{a}.\mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \cos \theta$, we now see that we need to calculate

$$\int_a^b \mathbf{F}.\frac{\mathbf{c}'(t)}{||\mathbf{c}'(t)||}||\mathbf{c}'(t)||dt = -\int_a^b \sqrt{2}||\mathbf{c}'(t)||dt.$$ 

Note that the negative is there because the angle between $\mathbf{c}'(t)$ and $\mathbf{F}$ is $\pi$. This last integral has value $-\sqrt{2} \cdot \text{arclength of } \partial S$. The arclength of $\partial S$ is $2\sqrt{6}\pi/3$ and the final answer is therefore $-4\sqrt{3}\pi/3$.

§8.4 #2. Let $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$. Compute $\iint_{\partial S} \mathbf{F}.d\mathbf{S}$ where $S$ is the ball of radius 1 (of course then $\partial S$, the boundary of $S$ is the unit sphere). By the Divergence theorem,

$$\iint_{\partial S} \mathbf{F}.d\mathbf{S} = \iiint_S \nabla \mathbf{F}.d\mathbf{V} = \iiint_S 3(x^2 + y^2 + z^2)d\mathbf{V} = \int_0^1 \int_0^{2\pi} \int_0^\pi 3p^4 \sin \phi d\phi d\theta dp = \frac{12\pi}{5}$$
§8.4 #7. Compute the flux of \( \mathbf{F} = (x - y^2)\mathbf{i} + y\mathbf{j} + x^3\mathbf{k} \) out of the rectangular solid \([0, 1] \times [1, 2] \times [1, 4]\).

**Solution.** Use the Divergence Theorem: \( \text{div}\mathbf{F} = 2 \). Therefore,

\[
\int \int_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \int \int \int_W 2 \, dV = 2(1 - 0)(2 - 1)(4 - 1) = 6.
\]

§8.4 #18. Suppose \( \mathbf{F} \) is tangent to the closed surface \( S = \partial W \) of a region \( W \). Prove that \( \iiint_W (\text{div}\mathbf{F}) \, dV = 0 \).

**Solution.** By Gauss divergence theorem, the above integral is equal to \( \iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iint_{\partial W} \mathbf{F} \cdot \mathbf{n} \, dS \).

Since \( \mathbf{F} \) is tangent to \( \partial W \), \( \mathbf{F} \cdot \mathbf{n} = 0 \), therefore the result follows.