book problems

§11.3 #4(b). Show there are infinitely may even abundant numbers. As per the hint, consider \( n = 2^k \cdot 3 \). \( \sigma(n) = (2^{k+1} - 1)4 \). We need to show that this is bigger than \( 2^{k+1} + 2^k \cdot 3 \). This follows because \( 2^{k+1} - 1 > 0 \) by our assumption that \( k > 1 \). Add \( 3 \cdot 2^{k-1} \) to both sides. We get \( 4 \cdot 2^{k-1} - 1 = 2^{k+1} - 1 > 3 \cdot 2^{k-1} \). Multiplying both sides by 4 we get \( \sigma(n) = (2^{k+1} - 1)4 > 3 \cdot 2^{k+1} \).

§11.4 #17(a). Note that \( F_n = 2^{2^n} + 1 \equiv 1 \) (mod 4) for all \( n \geq 2 \). Therefore, \( (3/F_n) = (F_n/3) \). Now, \( 2^{2^n} + 1 \equiv (-1)^{2^n} + 1 \equiv -1 \) (mod 3). Therefore \( (3/F_n) = (-1/3) = -1 \). To compute \( (5/F_n) \) note that \( 2^{2^n} + 1 = 4^{2^n-1} + 1 \equiv (-1)^{2^n-1} + 1 \equiv 1 + 1 \equiv 2 \) (mod 5). Therefore, \( (5/F_n) = (F_n/5) = (2/5) = -1 \). By problem 15, section 9.3, since \( F_n \) is of the form \( p = 2^{4n} + 1 \), we see that 7 is a primitive root of \( F_n \). Therefore, the order of 7 (mod \( F_n \)) is \( F_n - 1 \) and therefore \( 7^{(F_n-1)/2} \equiv -1 \) (mod \( F_n \)). (Since \( 7^{(F_n-1)/2} \) satisfies \( x^2 \equiv 1 \) (mod \( F_n \)) and \( 7^{(F_n-1)/2} \) is not equivalent to 1 (mod \( F_n \)), we must have \( 7^{(F_n-1)/2} \equiv -1 \) (mod \( F_n \)).

Non-book problems:

1. We proceed by induction. Case \( n = 0 \) comes down to showing \( 3 = F_0 = F_1 - 2 \). But this is true since \( F_1 = 5 \). Assume by induction \( \prod_{i=0}^{k-1} = F_k - 2 \). We need to show that \( \prod_{i=0}^{k} F_i = F_{k+1} - 2 \). By the induction hypothesis this boils down to showing \( (F_{k-1})F_k = F_{k+1} - 2 \) which follows quickly.

2. Let \( q \) be a prime dividing \( \prod_{i=0}^{n} F_i \). Claim: \( q \) does not divide \( F_{n+1} \). Proof of claim: If \( q \mid F_{n+1} \), then \( q \mid (F_{n+1} - F_0 F_1 \ldots F_n) \) and therefore \( q \mid 2 \) by part 1. This is a contradiction since all \( F_i \) are odd. Therefore \( q \nmid F_{n+1} \). Let \( T_n = \{ \text{primes dividing} \prod_{i=0}^{n} F_i \} \). Then each \( T_n \) is a proper subset of \( T_{n+1} \). In particular, as \( n \) goes to infinity, our list of primes is infinitely long.