Instructions. This project will be unpledged. It is an open-book, open-notes project. Feel free to discuss it with your colleagues, or to consult with the instructor. However, the write up is your job, and you should do that much without any help.

Your grade on the project will reflect your mathematical skills as well as your communication skills. You will be expected to prepare your project submission with the same care as you do a paper in any other course. The only concession we make is that it is not necessary to type your paper, because typing mathematics and making it attractive and readable is quite difficult.

In preparing your paper, you should try to use good mathematical and scientific style. You should model your presentation on what you see in your textbooks in mathematics and in application areas. Your write-up should be designed to communicate your results to the reader. Approximately 20% of your grade will be based on the effectiveness with which you have done this. This means that you should:

- Include a separate cover sheet as the first page of your project containing the project title, your name, your instructor’s name, and your section number.
- Organize your write-up into paragraphs and complete sentences.
- Number your figures and refer to them by number in your text so the reader can find them when needed.
- Number the formulas that you develop and then refer to.
- Number your pages.
- Give proper credit to your sources by referencing them in footnotes or in a bibliography.
- Be sure to give reasons for your statements. Your arguments should be like those in the book. Make your explanations as concise as possible.
- Allow yourself enough time to transform your figures, formulas, and findings into a readable, understandable write-up.
The Possum Plague: A model in the making

The goal of this project is to analyze most of the possibilities for a system modeling the population/disease dynamics of the population of possums in New Zealand. The description of the model follows the book "Differential Equations: a modeling with perspective" by R. Borelli and C. Coleman.

Figure 1. Australian possum (Trichosurus vulpecula)

General description. The ecological balance in New Zealand has been disturbed by the introduction of the Australian possum, a marsupial the size of a domestic cat with the proper name Trichosurus vulpecula. The animal was introduced in the 1830s for a planned fur trade. There were no natural predators in the forests of New Zealand, and the possum population rapidly increased. Today the estimated population is 70 million, and only a few areas are possum-free. The animals have become a reservoir of a type of tuberculosis with about half of the possum population infected. This poses a threat of transfer of the disease to the livestock that form an important part of the New Zealand economy. An intense effort is under way to understand possum ecology and the disease. This effort includes the construction of mathematical models for the possum/disease dynamics. The simplest of these models is treated here.

Modeling the population/disease dynamics Suppose that $P$ is the population of possums and $I$ is the subpopulation of infected possums, both measured in units of tens of millions. Note that $P \geq I$ and that the population of healthy possums is $P - I$. The simplest dynamical model for $P(t)$ and $I(t)$ is

\[
\begin{align*}
P' &= (a - b)P - \alpha I \\
I' &= \beta I(P - I) - (\alpha + b)I
\end{align*}
\] (1)

where time $t$ is measured in years, $a$ and $b$ are respective natural birth and death rate coefficients measured in units of year$^{-1}$, and $\alpha$ is the disease-induced death rate coefficient in year$^{-1}$ units. The number $\beta$ (measured in units of $(10^7 \cdot \text{year})^{-1}$) is the mass action coefficient that
measures the effectiveness of interactions in transmitting the disease from population $I$ of the diseased to the population $P - I$ of healthy animals. Once a possum contracts tuberculosis, it never recovers, so there is no population of “recovered”.

This nonlinear model is only a crude approximation to reality and neglects such factors as spatial variations in the population levels and differences due to age structure. But the model is quite effective in giving a macroscopic picture of the situation.

(a) The system for $P$ and $I$ has four parameters, $a, b, \alpha, \beta$. We can reduce the number of parameters and ease the analysis by introducing the dimensionless variables

$$x = \frac{\beta}{\alpha}P, \quad y = \frac{\beta}{\alpha}I, \quad \text{and} \quad s = \alpha t$$

Show that in terms of $x$ and $y$ the system (1) becomes

$$\frac{dx}{ds} = cx - y$$
$$\frac{dy}{ds} = y(x - y - r)$$

where the new constants are

$$c = \frac{a - b}{\alpha} \quad \text{and} \quad r = \frac{\alpha + b}{\alpha}.$$ 

Now we have only two parameters in the system (2). There is a reason for focusing on parameters $c$ and $r$. The most likely way to reduce the possum population is to reduce its birth rate or raise its natural death rate, that is, alter $a$ or $b$, and so change $c$ and $r$. Notice that the parameter $c$ can take any value, while $r$ must be positive.

(b) Show that under the assumption $c + r \geq 1$ the region $R$ of viable situations given by $0 \leq y \leq x$ is (forward) invariant. More precisely, show that an orbit starting at $(x_0, y_0)$ with $0 \leq y_0 \leq x_0$ stays in region $R$ as time increases. (Hint: it is necessary to analyze the vector field on the boundary lines of $R$; on the diagonal $y = x$ the vector field is $(cx - x, -rx)$.

From now on, we assume that the parameters $c$ and $r$ satisfy the relation $c + r \geq 1$.

(c) Show that if $c < 0$ then all orbits in $R$ tend to the origin. In other words, the possums die out (and the disease, presumably, with them). What is the meaning of $c < 0$?

(d) Study what happens in the case $c = 0$. How many equilibrium points are there? Study their stability (if possible). Explain the long term behavior of the possum population.

(e) Assume $r = 2$. Analyze the number of fixed points and their stability properties as parameter $c$ is varied. Can you pinpoint some values of $c$ where important changes in the behavior of the system happen? Explain the long term behavior of the population in each case.

(f) Find in the parameter semi-plane $cr$ $(c + r \geq 1)$ the curves that separate various long term behaviors. Describe what the long term behavior is on each of these parameter curves and within the regions bounded by them.

In your analysis you can use any of the techniques you have learned in the course. In particular, on your way to answering the question, use pplane to compute and display various aspects of the systems in the typical cases. This can be used to motivate and illustrate your arguments.