1. Consider a bacteria population \( N(t) \) modeled by the separable equation

\[
\frac{dN}{dt} = \frac{2}{1 + t} N(t).
\]

(a) Explain the practical meaning of the time-dependent growth rate \( 2/(1 + t) \).
(b) Find the population \( N(t) \) assuming that \( N(0) = N_0 \).
(c) Find the initial population \( N_0 \) so that \( N(99) = 100,000 \).

2. A tank initially contains 100 gal of water. A sugar-water solution containing 1 lb of sugar per gallon is poured into the tank at a rate of 2 gal/min and the mixture is drained from the tank at the same rate.

(a) If \( Q(t) \) denotes the amount of sugar in the tank at time \( t \) show that \( Q(t) \) satisfies the differential equation:

\[
Q'(t) = 2 - \frac{Q(t)}{100}.
\]

(b) Find the amount of sugar \( Q(t) \) in the tank at any time.
(c) Find the moment of time \( T \) when the concentration of sugar in the tank is \( 1/2 \) lb/gal. The final answer may involve a logarithm function.

3. Consider a logistic type population model with a threshold value to growth

\[
\frac{dN}{dt} = rN \left( 1 - \frac{N}{K} \right) \left( \frac{N}{K_0} - 1 \right)
\]

where the parameters \( r, K, K_0 \) are positive with \( K_0 < K \).

(a) Find and classify the equilibrium points for this model.
(b) Use qualitative analysis to discuss the long term behavior of the population.
(c) Explain the role of parameter \( K_0 \).

EXAM CONTINUES ON NEXT PAGE!
4. An intravenous administration of a drug can be described by two-compartment model, with compartment 1 representing the blood plasma and compartment 2 representing body tissue. The dynamics of evolution of the system are given by the system of differential equations:

\[
\begin{align*}
{x}'_1 &= -(K_1 + K_3)x_1 + K_2x_2 \\
{x}'_2 &= K_1x_1 - K_2x_2
\end{align*}
\]

with \(K_1, K_2, K_3\) positive parameters.

(a) Draw a schematic diagram that shows the compartments and the flows into and out of them.

(b) Solve this system of differential equations for the special case \(K_1 = 1, K_2 = K_3 = 2\) and with the initial conditions \(x_1(0) = 10, x_2(0) = 0\). What happens in the limit as \(t\) goes to infinity?

(c) Sketch the phase plane portrait for the case when \(K_1 = 1, K_2 = K_3 = 2\). Determine the type and stability of the equilibrium point \((0, 0)\).

5. Consider two interacting species with populations \(x(t)\) and \(y(t)\) such that the first species is a predator of the second and is not able to survive in the absence of the prey; the second species thrives on its own but has limited resources. Model the interactions between the species with a system of differential equations. You are not required to solve the system.