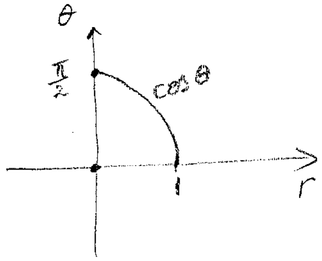
Math 212 - Hw #8

1b/5.4.

$$\int_0^{\pi/2}\int_0^{\cos\theta} \cos\theta \, dr\, d\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{\frac{\pi}{2}} = \boxed{\frac{\pi}{4}}$$



Change the order: $0 \le r \le 1$
$0 \le \theta \le \cos^{-1} r$;

The integral becomes $\int_0^1 \int_0^{\cos^{-1} r} \cos\theta \, d\theta\, dr = \int_0^1 \sin(\cos^{-1} r)\, dr$

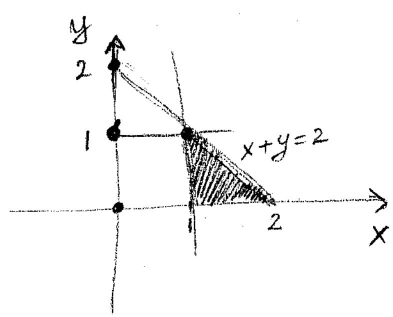
$$= \int_0^1 \sqrt{1-\cos(\cos^{-1} r)}\, dr = \int_0^1 \sqrt{1-r^2}\, dr$$

$$= \frac{r}{2}\sqrt{1-r^2} + \frac{1}{2}\arcsin r \Big|_0^1 = \boxed{\frac{\pi}{4}}.$$

1c/5.4

$$\int_0^1 \int_1^{2-y} (x+y)^2 \, dx\, dy = \int_0^1 \frac{1}{3}(x+y)^3\Big|_1^{2-y} dy$$

$$= \int_0^1 \frac{1}{3}\left(8-(y+1)^3\right) dy = \frac{8}{3}y - \frac{(y+1)^4}{12}\Big|_0^1 = \boxed{\frac{17}{12}}$$



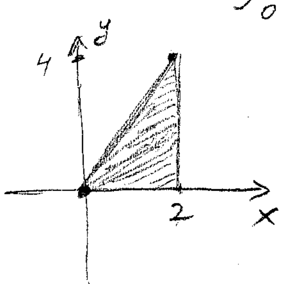
change the order: $1 \le x \le 2$
$0 \le y \le 2-x$

$$\int_1^2 \int_0^{2-x} (x+y)^2 \, dy\, dx = \int_1^2 \frac{1}{3}(8-x^3)\, dx$$

$$= \frac{8}{3}x - \frac{x^4}{12}\Big|_1^2 = \boxed{\frac{17}{12}}.$$

2c/5.4.

$$\int_0^4 \int_{y/2}^2 e^{x^2} dx\, dy \underset{\text{change the order}}{=} \int_0^2 \int_0^{2x} e^{x^2} dy\, dx$$



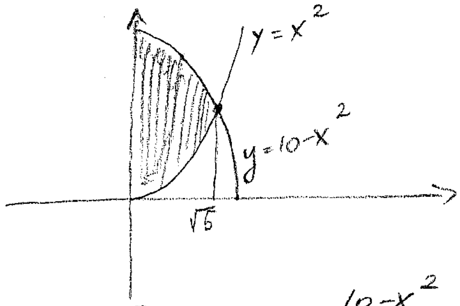
$$= \int_0^2 e^{x^2} y \Big|_0^{2x} dx = \int_0^2 e^{x^2} 2x\, dx = e^{x^2}\Big|_0^2 = \boxed{e^4 - 1}$$

3/5.4 Notice that $-1 \le \sin(x+y) \le 1$ for $x, y \in D$, hence

$$e^{-1} \le e^{\sin(x+y)} \le e, \text{ and}$$

$$\boxed{\frac{4\pi^2}{e}} = e^{-1}\cdot \text{area}(D) \le \iint_D e^{\sin(x+y)} \le e\cdot \text{area}(D) = \boxed{4\pi^2 \cdot e}$$

8/5.4.



The curves intersect at

$$x^2 = 10 - x^2 \Rightarrow 2x^2 = 10;\ x = \sqrt{5}$$
$(x>0)$.

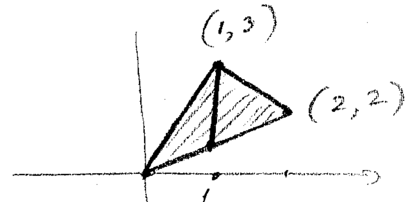
Hence, $\displaystyle\iint_D y^2\sqrt{x}\,dA = \int_0^{\sqrt5}\int_{x^2}^{10-x^2} y^2\sqrt{x}\,dy\,dx$

$$= \int_0^{\sqrt5} \frac{1}{3} y^3\sqrt{x}\,\Big|_{x^2}^{10-x^2} dx = \frac{1}{3}\int_0^{\sqrt5} (10-x^2)^3\sqrt{x} - x^6\sqrt{x}\,dx$$

$$= \frac{1}{3}\left(-\frac{4}{15}x^{15/2} + \frac{60}{11}x^{11/2} - \frac{600}{7}x^{7/2} + \frac{2000}{3}x^{3/2}\right)\Big|_0^{\sqrt5} =$$

$$= \frac{78800}{693}\cdot 5^{3/4}$$

10/5.4. Split the triangle into two parts, and



$$\iint_D e^{x-y}\,dx\,dy = \int_0^1\int_x^{3x} e^{x-y}\,dy\,dx + \int_1^2\int_x^{4-x} e^{x-y}\,dy\,dx$$

$$= \int_0^1 -e^{x-y}\Big|_x^{3x} dx + \int_1^2 -e^{x-y}\Big|_x^{4-x} dx$$

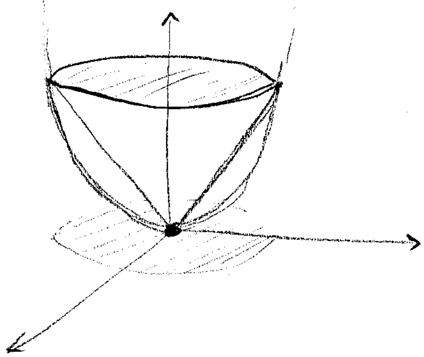
$$= \int_0^1 1 - e^{-2x}\,dx + \int_1^2 1 - e^{2x-4}\,dx = \frac{1}{2e^2} + \frac{1}{2} + \frac{1}{2e^2} + \frac{1}{2} = \boxed{1 + \frac{1}{e^2}}$$

4/5.5. $\displaystyle\int_0^1\int_0^1\int_0^1 z\cdot e^{x+y}\,dx\,dy\,dz = \int_0^1\int_0^1 z\cdot e^{x+y}\Big|_{x=0}^{x=1} dy\,dz$

$$= \int_0^1\int_0^1 \left(z\cdot e^{1+y} - z\cdot e^{y}\right)dy\,dz = \int_0^1 z\cdot(e^2 - e - e + 1)\,dz$$

$$= \boxed{\frac{1}{2}\left(e^2 - 2e + 1\right)}$$

$\boxed{5/5.5}$



The cone and the paraboloid intersect when $\left.\begin{array}{l} z=\sqrt{x^2+y^2} \\ z=x^2+y^2 \end{array}\right\} \to$

$z^2=z \Rightarrow z=0,\ z=1$

$z=1$ implies $x^2+y^2=1$.

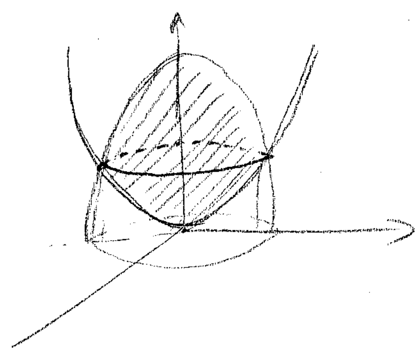
The region can be described as:

$$\boxed{\begin{array}{c} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \\ x^2+y^2 \le z \le \sqrt{x^2+y^2} \end{array}}$$

$\boxed{8/5.5}$ The region is in the first octant bounded by the vertical plane $x+y=4$ and by the plane $z=x+y+1$. Hence

$$\boxed{\begin{array}{c} 0 \le x \le 4 \\ 0 \le y \le 4-x \\ 0 \le z \le x+y+1. \end{array}}$$

$\boxed{9/5.5}$ The intersection of the two paraboloids is: $\left.\begin{array}{l} z=x^2+y^2 \\ z=10-x^2-2y^2 \end{array}\right\} \to$



$\Rightarrow 2x^2+3y^2=10$ (ellipse)

The region is described as: $-\sqrt{5} \le x \le \sqrt{5}$

$$-\sqrt{\frac{10-2x^2}{3}} \le y \le \sqrt{\frac{10-2x^2}{3}}$$

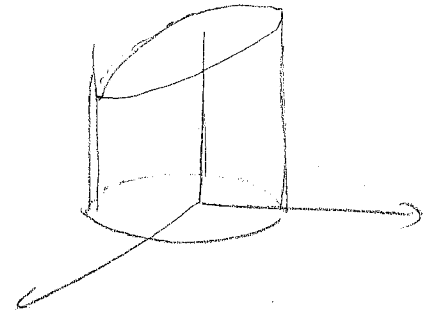
$$x^2+y^2 \le z \le 10-x^2-2y^2$$

$$\text{Volume} = \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{\frac{10-2x^2}{3}}}^{\sqrt{\frac{10-2x^2}{3}}} \int_{x^2+y^2}^{10-x^2-2y^2} 1\, dz\, dy\, dx = \ldots$$

$\boxed{10}$ $\boxed{5.5}$ $-1 \le y \le 1$

$-\sqrt{2-2y^2} \le x \le \sqrt{2-2y^2}$

$0 \le z \le \dfrac{2-x-y}{2}$



$$\text{Volume} = \int_{-1}^{1} \int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}} \int_{0}^{\frac{2-x-y}{2}} 1 \, dz \, dx \, dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}} \left(1 - \frac{x}{2} - \frac{y}{2}\right) dx \, dy$$

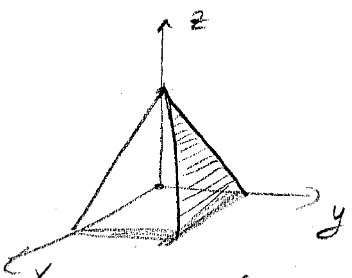
$$= \int_{-1}^{1} x(1-y/2) - \frac{x^2}{4} \Big|_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}} dy = \int_{-1}^{1} (2-y)\sqrt{2-2y^2} \, dy$$

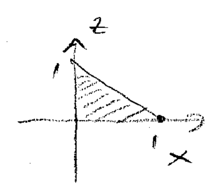
$$= 2\sqrt{2} \cdot \int_{-1}^{1} \sqrt{1-y^2} \, dy - \sqrt{2}\int_{-1}^{1} y\sqrt{1-y^2} \, dy$$

$$= 2\sqrt{2} \cdot \left(\frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\arcsin y\right)\Big|_{-1}^{1} + \sqrt{2}\,\frac{1}{3}(1-y^2)^{\frac{3}{2}} \Big|_{-1}^{1}$$

$$= \pi\sqrt{2} + 0 = \boxed{\pi\sqrt{2}}$$

$\boxed{19}$ $\boxed{5.5}$ The region can be described as:





$0 \le x \le 1$

$0 \le z \le 1-x$

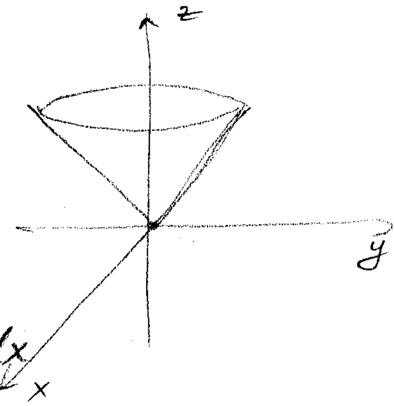
$0 \le y \le 1-z$ (because the plane passing through $(0,0,1)$ $(0,1,0)$, $(1,1,0)$ has equation $y+z=1$)

Thus $\displaystyle\int_{x=0}^{1} \int_{z=0}^{1-x} \int_{y=0}^{1-z} (1-z^2)\, dy \, dz \, dx = \boxed{\frac{3}{10}}$

23. 5.5 $-1 \le x \le 1$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$\sqrt{x^2+y^2} \le z \le 1.$



$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\,dz\,dy\,dx$$

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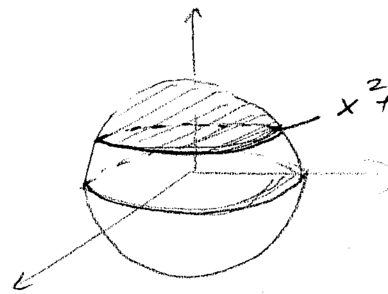
24. 5.5 $-\frac{\sqrt{3}}{2} \le x \le \frac{\sqrt{3}}{2}$

$-\sqrt{\frac{3}{4}-x^2} \le y \le \sqrt{\frac{3}{4}-x^2}$

$\frac{1}{2} \le z \le \sqrt{1-x^2-y^2}$



$$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}\int_{-\sqrt{\frac{3}{4}-x^2}}^{\sqrt{\frac{3}{4}-x^2}}\int_{\frac{1}{2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx.$$
