Math 211 - Hw #6 - #7

$\boxed{5/3.3}$ Find critical points: $\frac{\partial f}{\partial x} = 0$; $\frac{\partial f}{\partial y} = 0$.

$$\frac{\partial f}{\partial x} = 2x e^{1+x^2-y^2} = 0$$

$$\frac{\partial f}{\partial y} = -2y e^{1+x^2-y^2} = 0 \quad \Rightarrow \quad x=0,\ y=0.$$

Apply second derivative test to $(0,0)$:

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 2e^{1+x^2-y^2} + 4x^2 e^{1+x^2-y^2}\Big/_{(0,0)} = 2e$$

$$\frac{\partial^2 f}{\partial y^2}(0,0) = -2e^{1+x^2-y^2} + 4y^2 e^{1+x^2-y^2}\Big/_{(0,0)} = -2e$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = -4xy\, e^{1+x^2-y^2}\Big/_{(0,0)} = 0 = \frac{\partial^2 f}{\partial x \partial y}(0,0)$$

$$D = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial y \partial x}(0,0) \\ \frac{\partial^2 f}{\partial x \partial y}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{vmatrix} = -4e^2 < 0, \text{ hence}$$

<u>$(0,0)$ is a saddle point.</u>

$\boxed{7/3.3}$

$$\frac{\partial f}{\partial x} = 6x + 2y + 2 = 0$$

$$\frac{\partial f}{\partial y} = 2x + 2y + 1 = 0$$

$\Rightarrow$ critical point $\left(-\frac{1}{4}, -\frac{1}{4}\right)$

$$D = \begin{vmatrix} 6 & 2 \\ 2 & 2 \end{vmatrix} = 8 > 0, \text{ and } \frac{\partial^2 f}{\partial x^2} = 6 > 0, \text{ hence}$$

<u>$\left(-\frac{1}{4}, -\frac{1}{4}\right)$ is a local min.</u>

$\boxed{9/3.3.}$

$$\frac{\partial f}{\partial x} = -2x\sin(x^2+y^2) = 0$$

$$\Rightarrow \quad x=0,\ y=0 \text{ or } x^2+y^2 = \pi k,\ k\in\mathbb{Z}.$$

$$\frac{\partial f}{\partial y} = -2y\sin(x^2+y^2) = 0$$

Notice that we have an infinite number of critical points.

We study only $(0,0)$, $(\sqrt{\pi/2}, \sqrt{\pi/2})$ and $(0, \sqrt{\pi})$.

For $(0,0)$: $\frac{\partial^2 f}{\partial x^2}(0,0) = -2\sin(x^2+y^2) - 4x^2\cos(x^2+y^2)\Big|_{(0,0)} = 0$

Also $D = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$. This is a degenerate case. We cannot use the 2^{nd} derivative test.

But, looking at $f(x,y)$, notice that $f(x,y) = \cos(x^2+y^2) \le 1$ and $f(0,0) = 1$, hence <u>$(0,0)$ is a maximum point.</u>

For $(\sqrt{\pi/2}, \sqrt{\pi/2})$: $\frac{\partial^2 f}{\partial x^2}(\sqrt{\pi/2}, \sqrt{\pi/2}) = 2\pi$ and

$D = \begin{vmatrix} 2\pi & 2\pi \\ 2\pi & 2\pi \end{vmatrix} = 0$ - degenerate case.

Need to look at $f(x,y)$ about $(\sqrt{\pi/2}, \sqrt{\pi/2})$

$f(\sqrt{\pi/2}, \sqrt{\pi/2}) = \cos(\pi) = -1$, hence <u>$(\sqrt{\pi/2}, \sqrt{\pi/2})$ is a minimum point.</u>

For $(0, \sqrt{\pi})$: $\frac{\partial^2 f}{\partial x^2}(0, \sqrt{\pi}) = 0$

$D = \begin{vmatrix} 0 & 0 \\ 0 & 4\pi \end{vmatrix} = 0$ degenerate case

Since $f(0, \sqrt{\pi}) = \cos(\pi) = -1$ and $f(x,y) = \cos(x^2+y^2) \ge -1$, <u>$\Rightarrow (0, \sqrt{\pi})$ is a minimum point.</u>

$\boxed{17/3.3}$ $f(x,y) = (x^2+3y^2)e^{1-x^2-y^2}$

$$\frac{\partial f}{\partial x} = 2x e^{1-x^2-y^2}(1-x^2-3y^2) = 0$$

$$\frac{\partial f}{\partial y} = 2y e^{1-x^2-y^2}(3-x^2-3y^2) = 0$$

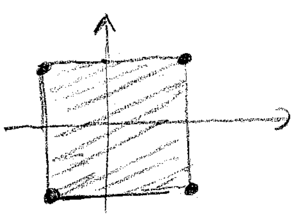
$\Rightarrow (0,-1); (0,1); (-1,0)$
$(1,0); (0,0)$.

Applying the second derivative test, we get:

Point	$\frac{\partial^2 f}{\partial x^2}$	$\frac{\partial^2 f}{\partial y \partial x}$	$\frac{\partial^2 f}{\partial y^2}$	D	Type
$(0,-1)$	-4	0	-12	48	local max
$(0,1)$	-4	0	-12	48	local max
$(-1,0)$	-4	0	4	-16	saddle
$(1,0)$	-4	0	4	-16	saddle
$(0,0)$	$2e$	0	$6e$	$12e^2$	local min

$\boxed{34/3.3.}$ The candidates are critical points inside the rectangle and on each side.

Inside the rectangle: $\frac{\partial f}{\partial x} = y = 0$, $\frac{\partial f}{\partial y} = x = 0$ $\rightarrow (0,0)$



For the side $x=-1$: $f(-1,y) = -y$; min at $y=1$; max at $y=-1$
For the side $y=1$: $f(x,1) = x$; min at $x=-1$; max at $x=1$
For the side $y=-1$: $f(x,-1) = -x$, min at $x=1$; max at $x=-1$
For the side $x=1$: $f(1,y) = y$, min at $y=-1$, max at $y=1$

Compare the values of f at $(0,0)$; $(-1,1)$; $(-1,-1)$; $(1,-1)$; $(1,1)$.
They are $f(0,0)=0$, $f(-1,1)=-1$; $f(-1,-1)=1$; $f(1,-1)=-1$; $f(1,1)=1$
hence $(1,1)$, $(-1,-1)$ are absolute maxima and $(1,-1)$, $(-1,1)$ abs. minima.

$\boxed{1/3.4}$ Need to solve $\nabla f = \lambda \nabla g$ $\quad f(x,y,z) = x - y + z$

$g(x,y,z) = x^2+y^2+z^2 (=2)$

This gives: $\begin{array}{l} 1 = \lambda 2x \\ 1 = -\lambda 2y \\ 1 = \lambda 2z \end{array} \Rightarrow x = \frac{1}{2\lambda},\ y = -\frac{1}{2\lambda},\ z = \frac{1}{2\lambda}$

Substitute into $x^2+y^2+z^2=2$: $\frac{3}{4\lambda^2} = 2 \Rightarrow \lambda = \pm \frac{\sqrt{3}}{2\sqrt{2}}$,

Hence, we get two critical points $\left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$ and $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\right)$.

The problem has absolute extrema, because the set on which we study their existence $(x^2+y^2+z^2=2)$ is bounded and closed.

Since $f\left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right) = 3\sqrt{\frac{2}{3}}$

$f\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\right) = -3\sqrt{\frac{2}{3}}$ we get

$\left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$ - absolute max. point

$\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\right)$ - absolute min. point

$\boxed{5/3.4.}$ Solve $\nabla f = \lambda \nabla g$, where $f(x,y) = 3x+2y$

$g(x,y) = 2x^2+3y^2 (=3)$

$\begin{array}{l} 3 = \lambda 4x \\ 2 = \lambda 6y \end{array} \Rightarrow x = \frac{3}{4\lambda},\ y = \frac{2}{6\lambda} = \frac{1}{3\lambda}$

Substitute into $2x^2+3y^2=3$: $2 \cdot \frac{9}{16\lambda^2} + 3 \cdot \frac{1}{9\lambda^2} = 3$

$\lambda^2 = \frac{1}{3}\left(\frac{9}{8} + \frac{1}{3}\right) = \frac{35}{72} \Rightarrow \lambda = \pm \frac{\sqrt{70}}{12}$

Two critical points: $\left(\frac{9}{\sqrt{70}}, \frac{4}{\sqrt{70}}\right)$; $\left(-\frac{9}{\sqrt{70}}, -\frac{4}{\sqrt{70}}\right)$

Compare the values of f at these points and get

$\left(\frac{9}{\sqrt{70}}, \frac{4}{\sqrt{70}}\right)$ - absolute max, $\left(-\frac{9}{\sqrt{70}}, -\frac{4}{\sqrt{70}}\right)$ - abs. min.

$\boxed{11/3.4.}$ Candidates are critical points inside the disk and on the boundary of it (unit circle)

Inside the disk: $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0 \Rightarrow (0,0)$.

On the boundary, we use Lagrange multipliers:

$$\nabla f = \lambda \nabla g \quad \text{where} \quad f(x,y) = x^2 + xy + y^2$$
$$g(x,y) = x^2 + y^2 \ (= 1).$$

$$\begin{array}{rl} & 2x + y = \lambda \cdot 2x \\ + & x + 2y = \lambda \cdot 2y \\ \hline & 3(x+y) = 2\lambda(x+y) \end{array} \Rightarrow \lambda = \frac{3}{2} \text{ or } x + y = 0.$$

If $\lambda = \frac{3}{2} \Rightarrow 2x + y = 3x \Rightarrow x = y$ and from $x^2+y^2=1$ we get $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

If $x + y = 0 \Rightarrow y = -x$ and from $x^2+y^2=1$ we get $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Compare the values of f:

$$f(0,0) = 0 \qquad f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{3}{2} = f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$
$$f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{1}{2} = f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

Hence $(0,0)$ is the absolute min point (and 0 is the min. value, $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ are the abs. max. points and $\frac{3}{2}$ is the max. value.

$\boxed{12/3.4}$ We need to maximize the volume $f(x,y,z) = xyz$ constrained to $g(x,y,z) = xy + 2xz + 2yz \,(= 16)$

Solve for $\nabla f = \lambda \nabla g$

$$yz = \lambda\,(y + 2z)$$
$$xz = \lambda\,(x + 2z)$$
$$xy = \lambda\,(2x + 2y)$$

Substract the first two equations:

$$z\,(y-x) = \lambda\,(y-x) \Rightarrow \lambda = z \text{ or } y - x = 0.$$

If $z = \lambda$: $\lambda y = \lambda y + 2\lambda^2 \Rightarrow \lambda = 0$ hence
$\lambda x = \lambda x + 2\lambda^2$ $\qquad xy = 0$ impossible
$xy = 2x\lambda + 2y\lambda$

If $x = y$: $\quad xz = \lambda x + 2\lambda z$

$$x^2 = \lambda \cdot 4x \Rightarrow x = 4\lambda,\; y = 4\lambda,\; z = 2\lambda$$

Substitute into $xy + 2xz + 2yz = 16$

$$16\lambda^2 + 16\lambda^2 + 16\lambda^2 = 16 \Rightarrow \lambda^2 = \frac{1}{3} \Rightarrow$$

$$\Rightarrow \lambda = \sqrt{\tfrac{1}{3}} \text{ (we need positive values for } x, y, z).$$

Hence $\underline{x = \frac{4}{\sqrt{3}},\; y = \frac{4}{\sqrt{3}},\; z = \frac{2}{\sqrt{3}}}$

This gives us the maximum volume to be $\underline{\frac{32}{3\sqrt{3}}}$.

$\boxed{28/3.4}$ We need to maximize $Q(x,y) = xy$ when $C(x,y) = 2x + 3y = 10$.

$\nabla Q = \lambda \nabla C$: $\quad y = 2\lambda$
$\qquad\qquad\qquad\quad x = 3\lambda \quad \Rightarrow 6\lambda + 6\lambda = 10 \Rightarrow \lambda = \frac{5}{6}$

and $\underline{x = \frac{5}{2},\; y = \frac{5}{3}}$

This gives the maximum quantity to be $\underline{\frac{25}{6}}$.

$\boxed{1a/5.2}$ $\int_0^1\int_0^1 x^3+y^2\,dx\,dy = \int_0^1 \frac{x^4}{4}+x\cdot y^2\Big|_0^1 dy$

$= \int_0^1 \frac{1}{4}+y^2\,dy = \frac{y}{4}+\frac{y^3}{3}\Big|_0^1 = \frac{1}{4}+\frac{1}{3} = \underline{\underline{\frac{7}{12}}}$

$\boxed{1b/5.2}$ $\int_0^1\left(\int_0^1 ye^{xy}dy\right)dx = \int_0^1 e^{xy}\Big|_0^1 dx = \int_0^1 e^{x}-1\,dx$

$= e^x - x\Big|_0^1 = e-1-(1-0) = \underline{\underline{e-2}}$

$\boxed{2c/5.2}$ $\int_0^1\int_0^1 \sin(x+y)\,dx\,dy = \int_0^1 -\cos(x+y)\Big|_0^1 dy$

$= \int_0^1 -\cos(1+y)+\cos(y)\,dy = -\sin(1+y)+\sin y\Big|_0^1$

$= \underline{\underline{\mathbf{2}\sin 1 - \sin 2}}$

$\boxed{3/5.2}$ $\int_0^1\int_0^1 xy\,dx\,dy = \int_0^1 \frac{x^2}{2}y\Big|_0^1 dy = \int_0^1 \frac{1}{2}y\,dy$

$= \frac{y^2}{4}\Big|_0^1 = \underline{\underline{\frac{1}{4}}}$

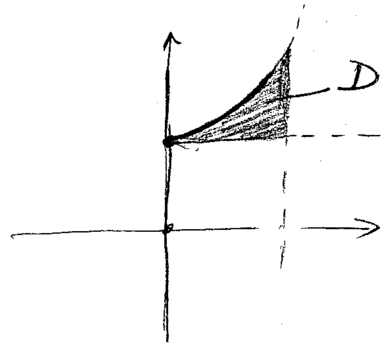
$\boxed{6/5.2}$ $\iint_D \sin y\,dA = \int_0^1\left(\int_0^{\pi/2}\sin y\,dy\right)dx = \int_0^1 -\cos y\Big|_0^{\frac{\pi}{2}}dx$

$= \int_0^1 1\,dx = x\Big|_0^1 = \underline{\underline{1}}$

$\boxed{1c/5.3}$ $\int_0^1\int_1^{e^x}(x+y)\,dy\,dx = \int_0^1 xy+\frac{y^2}{2}\Big|_1^{e^x}dx$

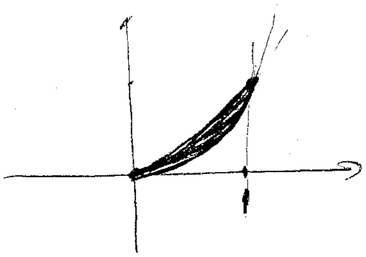
$= \int_0^1\left(xe^x+\frac{e^{2x}}{2}-x-\frac{1}{2}\right)dx = (x-1)e^x+\frac{e^{2x}}{4}-\frac{x^2}{2}-\frac{1}{2}x\Big|_0^1$

$= \underline{\underline{\frac{e^2-1}{4}}}$



1d/5.3

$$\int_0^1 \int_{x^3}^{x^2} y\,dy\,dx = \int_0^1 y^2/2 \Big|_{x^3}^{x^2} dx =$$



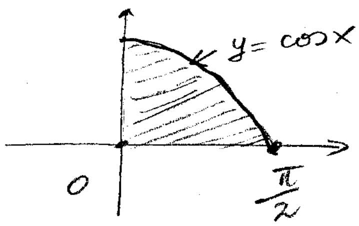
$$= \frac{1}{2}\int_0^1 (x^4 - x^6)\,dx = \frac{1}{2}\left(\frac{x^5}{5} - \frac{x^7}{7}\right)\Big|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) = \underline{\underline{\frac{1}{35}}}$$

2d/5.3

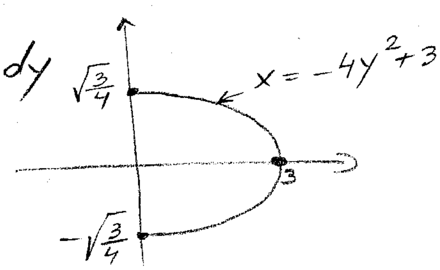
$$\int_0^{\pi/2}\int_0^{\cos x} y \sin x\,dy\,dx = \int_0^{\pi/2} \frac{y^2 \sin x}{2}\Big|_0^{\cos x} dx$$

$$= \int_0^{\pi/2} \frac{\cos^2 x \sin x}{2}\,dx = -\frac{1}{2}\,\frac{\cos^3 x}{3}\Big|_0^{\frac{\pi}{2}} = -0 + \frac{1}{6} = \underline{\underline{\frac{1}{6}}}.$$



7/5.3

$$\iint_D x^3 y\,dx\,dy = \int_{-\sqrt{3/4}}^{\sqrt{3/4}} \int_0^{-4y^2+3} x^3 y\,dx\,dy$$



$$= \int_{-\sqrt{3/4}}^{\sqrt{3/4}} \frac{x^4}{4} y \Big|_{x=0}^{x=-4y^2+3} dy$$

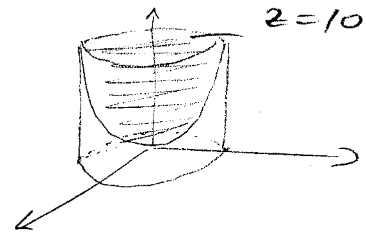
$$= \int_{-\sqrt{\frac{3}{4}}}^{\sqrt{3/4}} \frac{(-4y^2+3)^4}{4} y\,dy = \underline{\underline{0}}$$ because we integrate an odd function over a symmetric interval.

(or directly $= \dfrac{(-4y^2+3)^5}{20}\cdot\left(-\frac{1}{8}\right)\Big|_{-\sqrt{3/4}}^{\sqrt{3/4}} = 0$)

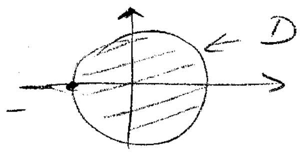
11/5.3

Volume = Volume (cylinder) −

− Volume under $z=x^2+y^2$



$$= \iint_D \left(10-(x^2+y^2)\right) dA$$



$$= \int_{-\sqrt{10}}^{\sqrt{10}} \int_{-\sqrt{10-x^2}}^{\sqrt{10-x^2}} 10-(x^2+y^2)\,dy\,dx$$

$$= \int_{-\sqrt{10}}^{\sqrt{10}} \left(10y - x^2y - \frac{y^3}{3}\right)\Bigg|_{-\sqrt{10-x^2}}^{\sqrt{10-x^2}} dx$$

$$= \int_{-\sqrt{10}}^{\sqrt{10}} 20\sqrt{10-x^2} - 2x^2\sqrt{10-x^2} - \frac{2}{3}\left(10-x^2\right)^{3/2} dx$$

$$= 20\cdot\left(\frac{x}{2}\sqrt{10-x^2} + \frac{10}{2}\arcsin\frac{x}{\sqrt{10}}\right)\Bigg|_{-\sqrt{10}}^{\sqrt{10}}$$

$$- 2\cdot\left(\frac{x}{8}\left(2x^2-10\right)\sqrt{10-x^2} + \frac{100}{8}\arcsin\frac{x}{\sqrt{10}}\right)\Bigg|_{-\sqrt{10}}^{\sqrt{10}}$$

$$- \frac{2}{3}\left(\frac{x}{8}\left(50-2x^2\right)\sqrt{10-x^2} + \frac{300}{8}\arcsin\frac{x}{\sqrt{10}}\right)\Bigg|_{-\sqrt{10}}^{\sqrt{10}}$$

$$= 100\pi - 2\,\frac{100}{8}\cdot\pi - \frac{2}{3}\,\frac{300}{8}\,\pi$$

$$= \underline{\underline{50\pi}}$$

We used the formulas: $\int\sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}$ $(a>0)$

$\int(a^2-x^2)^{3/2} = \frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2} + \frac{3a^4}{8}\arcsin\frac{x}{a}$ $(a>$

$\int x^2\sqrt{a^2-x^2} = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\arcsin\frac{x}{a}$ $(a>$