Math 212 - Hw #5

3/3.1 $f(x,y) = \cos(xy^2)$

$$\frac{\partial f}{\partial x} = -y^2\sin(xy^2) \qquad \frac{\partial f}{\partial y} = -2xy\sin(xy^2)$$

$$\frac{\partial^2 f}{\partial x^2} = -y^4\cos(xy^2)$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = -2y\sin(xy^2) - 2xy^3\cos(xy^2)$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = -2y\sin(xy^2) - 2xy^3\cos(xy^2)$$

Notice that the mixed derivative are equal since f is of class C^2

$$\frac{\partial^2 f}{\partial y^2} = -2x\sin(xy^2) - 2xy\cdot 2xy\cdot\cos(xy^2)$$
$$= -2x\sin(xy^2) - 4x^2y^2\cos(xy^2)$$

5/3.1 $f(x,y) = \dfrac{1}{\cos^2 x + e^{-y}}$

Writing $f(x,y) = (\cos^2 x + e^{-y})^{-1}$ makes things a bit easier.

$$\frac{\partial f}{\partial x} = (-1)(\cos^2 x + e^{-y})^{-2}(-2\cos x\sin x) = \frac{\sin 2x}{(\cos^2 x + e^{-y})^2}$$

$$\frac{\partial f}{\partial y} = (-1)(\cos^2 x + e^{-y})^{-2}\cdot(-e^{-y}) = \frac{e^{-y}}{(\cos^2 x + e^{-y})^2}$$

$$\frac{\partial^2 f}{\partial x^2} = 2\cos 2x\,(\cos^2 x + e^{-y})^{-2} + \sin 2x\cdot(-2)(\cos^2 x + e^{-y})^{-3}\cdot(-\sin 2x)$$
$$= \frac{2\cos 2x}{(\cos^2 x + e^{-y})^2} + \frac{2\sin^2 2x}{(\cos^2 x + e^{-y})^3} = \frac{2\cos 2x(\cos^2 x + e^{-y}) + 2\sin^2 2x}{(\cos^2 x + e^{-y})^3}$$

$$\frac{\partial^2 f}{\partial y^2} = -e^{-y}(\cos^2 x + e^{-y})^{-2} + e^{-y}(-2)(\cos^2 x + e^{-y})^{-3} \cdot (-e^{-y})$$

$$= -\frac{e^{-y}}{(\cos^2 x + e^{-y})^2} + \frac{2e^{-2y}}{(\cos^2 x + e^{-y})^3}$$

$$= \frac{-e^{-y}(\cos^2 x + e^{-y}) + 2e^{-2y}}{(\cos^2 x + e^{-y})^3} = \frac{e^{-2y} - e^{-y}\cos^2 x}{(\cos^2 x + e^{-y})^3}$$

$$\left(\text{or} \quad \frac{e^{-y} - \cos^2 x}{e^{y}(\cos^2 x + e^{-y})^3}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \sin 2x \cdot (-2)(\cos^2 x + e^{-y})^{-3} \cdot (-e^{-y}) = \frac{2\sin 2x}{e^{y}(\cos^2 x + e^{-y})^3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{-y}(-2)(\cos^2 x + e^{-y})^{-3} \cdot (-2\sin 2x) = \frac{2\sin 2x}{e^{y}(\cos^2 x + e^{-y})^3}$$

Again, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ (because f is of class C^2)

15b/3.1 $\quad f(x,y) = \cos\sqrt{x^2+y^2}$

Write $f(x,y) = \cos(x^2+y^2)^{\frac{1}{2}}$ for easier calculations

$$\frac{\partial f}{\partial x} = -\sin(x^2+y^2)^{\frac{1}{2}} \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x = -\frac{x\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y} = -\frac{y\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(-x\sin(x^2+y^2)^{\frac{1}{2}}(x^2+y^2)^{-\frac{1}{2}}\right) \quad \underline{\text{product rule}}$$

$$= -\sin(x^2+y^2)^{\frac{1}{2}}(x^2+y^2)^{-\frac{1}{2}} - x\cos(x^2+y^2)^{\frac{1}{2}} \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x \cdot (x^2+y^2)^{-\frac{1}{2}}$$

$$- x\sin(x^2+y^2)^{\frac{1}{2}} \cdot \left(-\frac{1}{2}\right)(x^2+y^2)^{-\frac{3}{2}} \cdot 2x$$

$$= -\frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} - \frac{x^2\cos\sqrt{x^2+y^2}}{(x^2+y^2)} + \frac{x^2\sin\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}}$$

Similarly, $\frac{\partial^2 f}{\partial y^2} = -\frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} - \frac{y^2\cos\sqrt{x^2+y^2}}{x^2+y^2} + \frac{y^2\sin\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}}$,

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} = \frac{-xy\cos\sqrt{x^2+y^2}}{x^2+y^2} + \frac{xy\sin\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}}.$$

$\boxed{19/3.1}$ One needs to check that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Indeed, $\frac{\partial u}{\partial x} = 3x^2 - 3y^2$ and $\frac{\partial^2 u}{\partial x^2} = 6x$

$\frac{\partial u}{\partial y} = -6xy$ and $\frac{\partial^2 u}{\partial y^2} = -6x$

hence the Laplace equation is verified.

$\boxed{2/3.2}$ $f(x,y) = \frac{1}{x^2+y^2+1}$

Notice that $\frac{\partial f}{\partial x}(0,0) = -\frac{2x}{(x^2+y^2+1)^2}\Big|_{(0,0)} = 0$; $\frac{\partial f}{\partial y}(0,0) = \frac{-2y}{(x^2+y^2+1)^2}\Big|_{(0,0)} = 0$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = -2(x^2+y^2+1)^{-2} - 2x(-2)(x^2+y^2+1)^{-3}\cdot 2x\Big|_{(0,0)} = -2$$

$$\frac{\partial^2 f}{\partial y^2}(0,0) = -2(x^2+y^2+1)^{-2} - 2y\cdot(-2)(x^2+y^2+1)^{-3}\cdot 2y\Big|_{(0,0)} = -2$$

$$\frac{\partial^2 f}{\partial x\partial y}(0,0) = 4y(x^2+y^2+1)^{-3}\cdot 2x\Big|_{(0,0)} = 0 \quad \left(= \frac{\partial^2 f}{\partial y\partial x}\right)$$

Taylor formula: $\boxed{f(h_1,h_2) = 1 - h_1^2 - h_2^2 + R_2(h_1,h_2)}$

($1-h_1^2-h_2^2$ is the second order Taylor approximation around $(0,0)$)

$\boxed{5/3.2.}$ $f(x,y) = \sin(xy) + \cos(xy)$ $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = y\cos(xy) - y\sin(xy)\Big|_{(0,0)} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = x\cos(xy) - x\sin(xy)\Big|_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = -y^2\sin(xy) - y^2\cos(xy)\Big|_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial y^2}(0,0) = -x^2\sin(xy) - x^2\cos(xy)\Big|_{(0,0)} = 0.$$

$$\frac{\partial^2 f}{\partial y\,\partial x}(0,0) = \cos xy - xy\sin(xy) - \left(\sin(xy) + yx\cos(xy)\right)\Big|_{(0,0)} = 1$$

Also, $f(0,0) = 1$. $\left(= \frac{\partial^2 f}{\partial x \partial y}(0,0)\right)$

Hence Taylor's formula is: $\boxed{f(h_1, h_2) = 1 + h_1 h_2 + R(h_1, h_2)}$

where $\displaystyle\lim_{(h_1,h_2)\to(0,0)} \frac{R(h_1,h_2)}{\|(h_1,h_2)\|^2} = 0$

(The second order Taylor approx about $(0,0)$ is $1 + h_1 h_2$)