Math 212 - Hw #4

7/2.4. $\bar{r}(t) = (\cos^2 t,\ 3t - t^3,\ t)$

Velocity vector: $\underline{\bar{r}'(t) = (-2\cos t \sin t,\ 3 - 3t^2,\ 1)}$

15/2.4. At $t=1$ the position is $(\sin 3, \cos 3, 2)$ and the tangent direction is given by $\bar{c}'(1) = (3\cos 3t, -3\sin 3t, 5t^{3/2})$ for $t=1$

$= (3\cos 3, -3\sin 3, 5)$.

Hence the tangent line is given by:

$$\boxed{\bar{l}(t) = (\sin 3, \cos 3, 2) + (t-1)(3\cos 3, -3\sin 3, 5)}$$

Remark: The parameter t was chosen so that when $t=1$ the line starts at $(\sin 3, \cos 3, 2)$, rather than when $t=0$.

19/2.4 The equation of the tangent line is:

$$\bar{l}(t) = \bar{c}(0) + (t-0)\cdot \bar{c}'(0)$$

$$= (4,0,1) + t\cdot(4,0,0)$$

At time $t=1$, the particle will be at $\underline{\bar{l}(1) = (8,0,1)}$.

56/2.5 $$\frac{d}{dt}(f\circ c) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= ye^{xy}\cdot 6t + xe^{xy}\cdot 3t^2$$

$$= t^3 \cdot e^{3t^5}\cdot 6t + 3t^2 e^{3t^5}\cdot 3t^2$$

$$= \underline{e^{3t^5}\cdot 15t^4}.$$

9/2.5 $D(f\circ g)(1,1) = Df\big|_{g(1,1)} \cdot Dg\big|_{(1,1)}$ (chain rule).

$$Df(u,v) = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} \frac{1}{\cos^2(u-1)} & -e^{v} \\ 2u & -2v \end{bmatrix}$$

Since $g(1,1) = (1,0)$: $Df(g(1,1)) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$

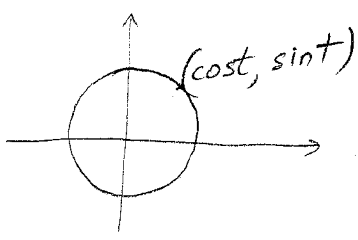
$$Dg(x,y) = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix} = \begin{bmatrix} e^{x-y} & -e^{x-y} \\ 1 & -1 \end{bmatrix}$$

So $D(f\circ g)(1,1) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \underline{\begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}}$.

$$(f\circ g)(x,y) = f(g(x,y)) = f(e^{x-y}, x-y)$$
$$= \underline{\left(\tan(e^{x-y}-1) - e^{x-y},\ e^{2(x-y)} - (x-y)^2\right)}.$$

Remark. Notice that a direct differentiation of $f\circ g$ would be much more complicated than the chain rule.

13/2.5.



Chain rule: $\dfrac{dT}{dt} = \dfrac{\partial T}{\partial x}\cdot\dfrac{dx}{dt} + \dfrac{\partial T}{\partial y}\cdot\dfrac{dy}{dt}$

$$= (2xe^{y} - y^3)\cdot(-\sin t) + (x^2\cdot e^{y} - 3xy^2)\cos t$$

$$= (2\cos t\, e^{\sin t} - \sin^3 t)(-\sin t) + (\cos^2 t\, e^{\sin t} - 3\cos t \sin^2 t)\cdot\cos t$$

$$= \underline{-2\cos t \sin t\, e^{\sin t} + \sin^4 t + \cos^3 t\, e^{\sin t} - 3\cos^2 t \sin^2 t}$$

Direct differentiation: $T(t) = \cos^2 t \cdot e^{\sin t} - \cos t \sin^3 t$

and $\underline{\dfrac{dT}{dt} = -2\cos t \sin t\, e^{\sin t} + \cos^3 t \cdot e^{\sin t} + \sin^4 t - 3\cos^2 t \sin^2 t}$

2d/2.6. $D_{\bar v} f(4,-2) = \nabla f(4,-2) \cdot \bar v$

$$= (y^2 + 3x^2 y,\ 2xy + x^3)\Big/_{(4,-2)} \cdot \left(\frac{1}{\sqrt{10}}\bar i + \frac{3}{\sqrt{10}}\bar j\right)$$

$$= (-92,\ 48)\cdot\left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

$$= \underline{\frac{52}{\sqrt{10}}}.$$

4a/2.6. The tangent plane is given by:

$$\nabla f(1,2,\tfrac{1}{3}) \cdot (x-1,\ y-2,\ z-\tfrac{1}{3}) = 0 \ldots$$

$\nabla f = (2x+3z,\ 4y,\ 3x)$ hence $\nabla f(1,2,\tfrac{1}{3}) = (3,8,3)$.

Plane: $3(x-1) + 8(y-2) + 3(z-\tfrac{1}{3}) = 0$, or

$$\underline{3x + 8y + 3z - 20 = 0.}$$

9/2.6. The surface is defined implicitly by $\cos(xy) - e^z = 2$.

A normal vector at $(1,\pi,0)$ is given by $\nabla f(1,\pi,0)$,

where $f(x,y,z) = \cos(xy) - e^z$.

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (-y\sin(xy), -x\sin(xy), -e^z)$$

$$\nabla f(1,\pi,0) = (0,0,-1) = -\bar k$$

This is already a unit vector, hence $\underline{-\bar k \text{ (or } \bar k)}$ is a unit normal to the surface

16/2.6. $T(x,y,z) = e^{-x^2-2y^2-3z^2}$

(a) He should proceed in the direction $-\nabla T(1,1,1)$.

$$\nabla T(x,y,z) = \left(-2x e^{-x^2-2y^2-3z^2}, -4y e^{-x^2-2y^2-3z^2}, -6z e^{-x^2-2y^2-3z^2}\right)$$

$$\nabla T(1,1,1) = (-2e^{-6}, -4e^{-6}, -6e^{-6})$$

and $-\nabla T(1,1,1) = \underline{(2e^{-6}, 4e^{-6}, 6e^{-6})}$.

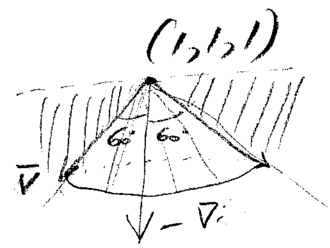
(b) The temperature will decrease by $\| e^{8}(2e^{-6}, 4e^{-6}, 6e^{-6})\|$
$= \|(2e^{2}, 4e^{2}, 6e^{2})\| = \sqrt{4e^4+16e^4+36e^4} = \underline{\underline{2\sqrt{14}\, e^{2}}}$

(c) The direction of fastest decrease is not viable, so let $\bar{v}$ be a possible vector direction. We want

$$\| -\nabla T \cdot \bar{v} \| \le \sqrt{14}\, e^{2}, \text{ or}$$

$$\underbrace{\|\nabla T\|}_{2\sqrt{14}\, e^{2}} \cdot \underbrace{\|\bar{v}\|}_{1} \cos\theta \le \sqrt{14}\, e^{2}. \text{ Thus}$$

$$\underline{\cos\theta \le \frac{1}{2}}$$



The angle θ between $\bar{v}$ and $-\nabla T$ should be $\geq 60°$ in order to have a rate of change $\le \sqrt{14}\, e^{2}$, and $\theta < 90°$ for a decrease.

The exact rate of change $\sqrt{14}\, e^{2}$ is obtained if $\underline{\underline{\theta = 60°}}$.

This gives a set of directions <u>situated on a cone</u>.

<u>Remark.</u> If $60° < \theta < 90°$, then the rate of change is $< \sqrt{14}\, e^{2}$.