Math 212 - Hw 14

1/8.1. $\int_C y\,dx - x\,dy = \iint_D -1-1\,dA = \int_{-1}^{1}\int_{-1}^{1} (-2)\,dx\,dy = \boxed{-8}$

3/8.1. Need to check $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

Line integral: path C: $\bar{c}(t) = (R\cos t, R\sin t)$ $\quad 0 \le t \le 2\pi$

$$\int_C P\,dx + Q\,dy = \int_0^{2\pi} R\cos t\, R^2\sin^2 t(-R\sin t) - R\sin t\, R\cos^2 t\, R\cos t\,dt$$

$$= \int_0^{2\pi} -R^4 \sin^3 t\cos t - R^4\cos^3 t\sin t\,dt$$

$$= \left(-R^4 \cdot \frac{\sin^4 t}{4} + R^4\frac{\cos^4 t}{4}\right)\Big|_0^{2\pi} = \boxed{0}$$

Double integral: (use polar coordinates):

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (-2xy - 2xy)\,dx\,dy$$

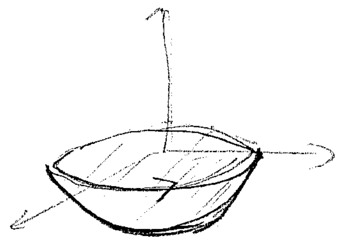
$$= \int_0^R \int_0^{2\pi} -4r^2 \sin\theta\cos\theta \cdot r\,d\theta\,dr$$

$$= \int_0^R -4r^3 \cdot \frac{\sin^2\theta}{2}\Big|_0^{2\pi} = \boxed{0}$$

13/8.1. $\int_C (y^2+x^3)\,dx + x^4\,dy = \iint_D (4x^3 - 2y)\,dx\,dy =$

$$= \int_0^1\int_0^1 (4x^3 - 2y)\,dx\,dy = \boxed{0}$$

1/8.2. $\iint_S (\nabla \times \bar{F})\cdot d\bar{S} = \int_C \bar{F}\cdot d\bar{s}$



where C is parametrized by:

$$\bar{c}(t) = (\cos t, \sin t, 0) \quad 0 \le t \le 2\pi$$

Hence $\iint_S (\nabla \times \bar{F})\cdot dS = \int_0^{2\pi} \bar{F}(\bar{c}(t))\cdot \bar{c}'(t)\,dt$

$$= \int_0^{2\pi} (\sin t\,\bar{i} - \cos t\,\bar{j} + 0\bar{k})\cdot(-\sin t\,\bar{i} + \cos t\,\bar{j} + 0\cdot\bar{k})\,dt$$

$$= \int_0^{2\pi} -1\,dt = \boxed{-2\pi}$$

5/8.2. We use Stokes' theorem and notice that the boundary curve of S is the circle $x^2+y^2=1$, $z=0$. There is no need to split the integral. As above:

$$\iint_S (\nabla \times \bar{F})\cdot d\bar{S} = \iint_C \bar{F}\cdot d\bar{s} = \int_0^{2\pi} (\cos t\,\bar{i} + \sin t\,\bar{j} + 0\,\bar{k})\cdot$$

$$\cdot\,(-\sin t\,\bar{i} + \cos t\,\bar{j} + 0\,\bar{k})\,dt = \int_0^{2\pi} 0\,dt = \boxed{0.}$$

10/8.2 Since the ellipsoid is a closed surface, it has no boundary curve $\Rightarrow \iint_S (\nabla \times \bar{F})\,dS = \int_C \bar{F}\cdot d\bar{s} = 0 \quad (C = \phi)$

25/8.2 Compute $\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y & -xz & -yz^2 \end{vmatrix} = (x-z^2)\bar{i} + 0\bar{j} + (-z-3)\bar{k}$

Direct computation of $\iint_S \nabla\times\bar{F}\cdot d\bar{S}$.

Notice that the surface S is the graph of $z = \frac{1}{2}(x^2+y^2)$ with $x^2+y^2\le 4$. Hence $d\bar{S} = (-\frac{\partial z}{\partial x}i - \frac{\partial z}{\partial y}j + \bar{k})\,dx\,dy$

$$\iint_S \nabla\times\bar{F}\cdot d\bar{S} = \iint_D \left[(x-z^2)\bar{i} + 0\bar{j} + (-z-3)\bar{k}\right]\cdot\left[-x\bar{i} - y\bar{j} + k\right] dx\,dy$$

$$= \iint_D -x^2 + x\cdot\left[\tfrac{1}{2}(x^2+y^2)\right]^2 - \tfrac{1}{2}(x^2+y^2) - 3 \; dx\,dy$$

$$\overset{\text{polar}}{=} \int_0^2\int_0^{2\pi}\left(-r^2\cos^2\theta + r\cos\theta\cdot\tfrac{1}{4}\cdot r^4 - \tfrac{1}{2}r^2 - 3\right) r\,d\theta\,dr$$

$$= \int_0^2 \left(-r^3\cdot\tfrac{1}{2}(\theta+\sin\theta\cos\theta)\Big|_0^{2\pi} - 0 - \pi r^3 - 6r\pi\right)dr = \int_0^2(-2\pi r^3 - 6\pi r)dr$$

$$= \boxed{-20\pi}$$

Using Stokes' theorem: $\int_C \bar{F}\cdot d\bar{s}$: $c(t) = (2\cos t, 2\sin t, 2)$ $\quad 0\le t\le 2\pi$.

$$\int_C F\cdot d\bar{s} = \int_0^{2\pi}(6\sin t\,\bar{i} - 4\cos t\,j - 8\sin^2 t)\cdot(-2\sin t\,\bar{i} + 2\cos t\,\bar{j} + 0k)\,dt$$

$$= \int_0^{2\pi} -12\sin^2 t - 8\cos^2 t\;dt$$

$$= \int_0^{2\pi}(-8 - 4\sin^2 t)\,dt = -16\pi - 2(t - \sin t\cos t)\Big|_0^{2\pi}$$

$$= -16\pi - 4\pi = \boxed{-20\pi.}$$

If one chooses the opposite orientation for the surface and for the boundary curve, the result is 20π.

3/8.3. Look for $f(x,y,z)$ such that:

$$\frac{\partial f}{\partial x} = 2xyz + \sin x\,; \quad \frac{\partial f}{\partial y} = x^2 z\,; \quad \frac{\partial f}{\partial z} = x^2 y$$

First relation $\Rightarrow$ $f(x,y,z) = x^2yz - \cos x + g(y,z)$

Plug into second $\Rightarrow$ $x^2 z + \frac{\partial g}{\partial y} = x^2 z \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$

Plug into third $\Rightarrow$ $x^2 y + h'(z) = x^2 y \Rightarrow h = \text{constant}$

Hence $f(x,y,z) = x^2 y z - \cos x + c$.

4/8.3. F - being conservative $\Rightarrow \int_C F \cdot d\bar{s} = f(c(\pi)) - f(c(0))$

$= f(-1, 0, \pi^4) - f(1,0,0) = (0 - \cos(-1)) - (0 - \cos 1) = \boxed{0}$

7/8.3. Since $\text{curl}\,\bar{F} = \nabla \times \bar{F} = -x\bar{K} \neq \bar{0}$, $\bar{F}$ is not conservative, so there does not exist a potential f.

15a) Since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ (test for conservative fields in $\mathbb{R}^2$)

$(3x^2 + 2xy = 2xy + 3x^2)$

the vector field $\bar{F}$ is conservative. A potential for $\bar{F}$ can be easily constructed (as in problem #3) to be

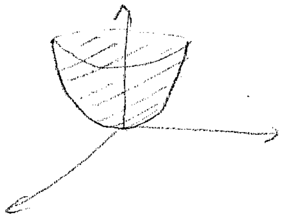
$$f(x,y,z) = \frac{1}{2}x^2y^3 + x^3y + c$$

Thus $\int_C F \cdot d\bar{s} = f(3,0) - f(1,1) = \boxed{-\frac{3}{2}}$

1/8.4. $\text{Flux} = \iint_S \bar{F} \cdot d\bar{S} = \iiint_W \operatorname{div} \bar{F} \, dV = \iiint_W 3 \, dV$

$= 3 \cdot \text{volume}(W) = 3 \cdot \frac{4}{3}\pi = \boxed{4\pi}$

5a/8.4

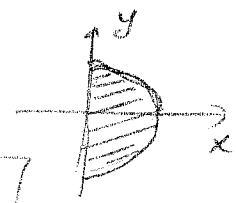


$\iint_{\partial W} \bar{F} \cdot d\bar{S} = \iiint_W \operatorname{div} \bar{F} \, dV$

$= \iiint_W x \, dx\, dy\, dz = \int_0^1 \int_0^{2\pi} \int_{r^2}^1 r\cos\theta \, r \, dz\, dr\, d\theta$

$= \int_0^1 \int_0^{2\pi} (1-r^2) r^2 \cos\theta \, d\theta \, dr = \boxed{0}$

5b/8.4 W is half of the paraboloid, so $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



$\iint_{\partial W} \bar{F} \cdot d\bar{S} = \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-r^2) r^2 \cos\theta \, d\theta \, dr = \boxed{\frac{4}{15}}$

7/8.4 $\text{Flux} = \iint_{\partial W} \bar{F} \cdot d\bar{S} = \iiint_W \operatorname{div} \bar{F} \, dV = \int_0^1 \int_1^2 \int_1^4 2 \, dz\, dy\, dx$

$= \boxed{6}$