Math 212 - Hw #13

1/7.4 Area $= \iint_S dS = \iint_D \| T_\varphi \times T_\theta \| \, d\varphi \, d\theta$

$$T_\varphi \times T_\theta = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi \\ -\sin\theta\sin\varphi & \cos\theta\sin\varphi & 0 \end{vmatrix}$$

$$= (\cos\theta\sin^2\varphi)\bar{i} + (\sin\theta\sin^2\varphi)\bar{j} + (\sin\varphi\cos\varphi)\bar{k}$$

and $\| T_\varphi \times T_\theta \| = \sin\varphi$.

Hence Area $= \int_0^{2\pi}\int_0^{\pi} \sin\varphi \, d\varphi \, d\theta = \boxed{4\pi}$

9/7.4

$x = a\cos\theta\sin\varphi$

$y = b\sin\theta\sin\varphi \qquad 0 \le \theta \le 2\pi \; ; \; 0 \le \varphi \le \pi.$

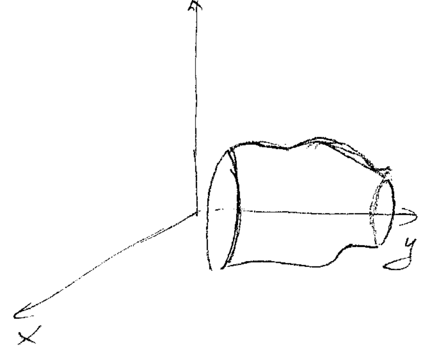
$z = c\cos\varphi$

$$T_\varphi \times T_\theta = (bc\cos\theta\sin^2\varphi)\bar{i} + (ac\sin\theta\sin^2\varphi)\bar{j} + (ab\sin\varphi\cos\varphi)\bar{k}$$

$$\| T_\varphi \times T_\theta \| = \sqrt{b^2c^2\cos^2\theta\sin^4\varphi + a^2c^2\sin^2\theta\sin^4\varphi + a^2b^2\sin^2\varphi\cos^2\varphi}$$

$$\text{Area} = \int_0^{2\pi}\int_0^{\pi} \| T_\varphi \times T_\theta \| \, d\varphi \, d\theta.$$

10/7.4



Parametrization:

$x = u\cos v$ $\quad a \le u \le b$

$y = f(u)$ $\quad 0 \le v \le 2\pi$

$z = u\sin v$

$$T_u \times T_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos v & f'(u) & \sin v \\ -u\sin v & 0 & u\cos v \end{vmatrix} = (uf'(u)\cos v)\bar{i} - u\bar{j} + (uf'(u)\sin v)\bar{k}$$

$$\| T_u \times T_v \| = |u| \sqrt{1 + (f'(u))^2}$$

Hence $\text{Area} = \int_a^b \int_0^{2\pi} |u| \sqrt{1+f'(u)^2}\, dv\, du$

$$\boxed{= 2\pi \int_a^b |u| \sqrt{1+f'(u)^2}\, du.}$$

11/7.4 $\text{Area} = 2\pi \int_0^1 |x| \sqrt{1+f'(x)^2}\, dx$

$= 2\pi \int_0^1 x\sqrt{1+4x^2}\, dx$

$= 2\pi \left(1+4x^2\right)^{3/2} \cdot \frac{1}{12}\Big|_0^1 = \boxed{\frac{\pi}{6}(5\sqrt{5}-1).}$

13/7.4 The surface is $z = 1-x-y$ with $x^2+2y^2 \le 1$.
A natural parametrization is obtained by using x, y as variables: $\| T_x \times T_y \| = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{3}.$

$$\text{Area} = \int_{x=-1}^{1} \int_{-\sqrt{\frac{1-x^2}{2}}}^{\sqrt{(1-x^2)/2}} \sqrt{3}\, dy\, dx = \sqrt{6}\int_{-1}^{1} \sqrt{1-x^2}\, dx = \boxed{\frac{\pi\sqrt{6}}{2}.}$$

3/7.5) One can use spherical coordinates or treat the hemisphere as the graph of $z=\sqrt{a^2-x^2-y^2}$.

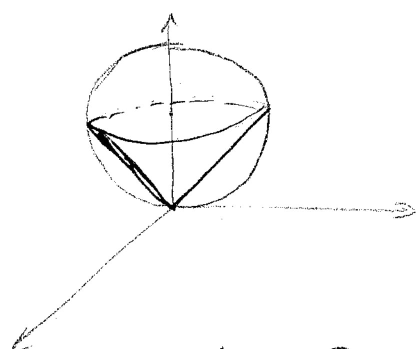
With spherical coordinates:

$$x = a\cos\theta\sin\varphi \qquad 0\le\theta\le 2\pi$$
$$y = a\sin\theta\sin\varphi \qquad 0\le\varphi\le\frac{\pi}{2},$$
$$z = a\cos\varphi$$

$$\|T_\varphi\times T_\theta\| = a^2\sin\varphi$$

$$\iint_S z\,dS = \int_0^{2\pi}\int_0^{\pi/2} a\cos\varphi\, a^2\sin\varphi\, d\varphi\, d\theta$$
$$= 2\pi a^3\cdot \frac{\sin^2\varphi}{2}\Big|_0^{\pi/2} = \boxed{\pi a^3.}$$

5a/7.5)



The equation of the sphere $x^2+y^2+z^2=2Rz$ can be rewritten as $x^2+y^2+(z-R)^2=R^2$ hence the sphere has center $(0,0,R)$ and radius R.

The sphere and the cylinder intersect when $2z^2=2Rz$, or $z=R$ $(x^2+y^2=R^2)$

The cone can be written as the graph of $z=\sqrt{x^2+y^2}$.

$$\text{Area} = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy \qquad (x^2+y^2\le R^2)$$
$$= \iint_D \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dx\,dy = \iint_D \sqrt{2}\,dx\,dy$$
$$= \sqrt{2}\,\text{area}(D) = \boxed{\sqrt{2}\cdot\pi R^2}$$

<u>Remark</u>: No need to use cylindrical coordinates.

7/7.5) S is the graph of $z = x^2+y^2$ so

$$\iint_S z\, dS = \iint_D (x^2+y^2)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dx\,dy \quad (x^2+y^2 \le 1)$$

$$= \iint_D (x^2+y^2)\sqrt{1+4(x^2+y^2)}\, dx\,dy$$

$$\overset{\text{polar}}{\underset{\text{coord}}{=}} \int_0^1 \int_0^{2\pi} r^2\sqrt{1+4r^2}\cdot r\, d\theta\, dr =$$

$$= 2\pi \int_0^1 r^3\sqrt{1+4r^2}\, dr = 4\pi \int_0^1 r^3\sqrt{r^2+\left(\tfrac{1}{2}\right)^2}\, dr$$

$$\overset{\text{formula}}{\underset{69}{=}} 4\pi\left(\frac{1}{5}r^2 - \frac{2}{15}\cdot\frac{1}{4}\right)\sqrt{\left(r^2+\frac{1}{4}\right)^3}\,\Big|_0^1$$

$$\boxed{= \pi\left(\frac{5\sqrt{5}}{12} + \frac{1}{60}\right).}$$

1/7.6) heat flow given by $-k\nabla T = -6x\bar{i} - 6z\bar{k}$.

The surface $x^2+z^2=2$ has normal $2x\bar{i}+2z\bar{k}$

and unit normal $\bar{n} = \frac{1}{2\sqrt{2}}(2x\bar{i}+2z\bar{k}) = \frac{1}{\sqrt{2}}(x\bar{i}+z\bar{k})$

Now, $$\iint_S (-k\nabla T)\cdot d\bar{S} = \iint_S (-k\nabla T)\cdot\bar{n}\, dS$$

$$= \iint_S \frac{1}{\sqrt{2}}\underbrace{(-6x^2-6y^2)}_{=-12}\, dS = -\frac{6}{\sqrt{2}}\iint_S 1\, dS$$

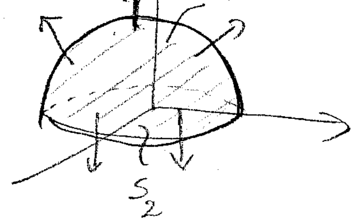
$$= -\frac{12}{\sqrt{2}}\ \text{area}(S) \quad \text{(where } S \text{ is a cylinder of length 2)}$$

$$= -\frac{12}{\sqrt{2}}\, 2\pi\sqrt{2}\cdot 2 = \boxed{-48\pi}$$

Remark: Notice no need to use cylindrical coordinate.

The flux is $\pm 48\pi$, depending on orientation.

3/7.6



We evaluate $\iint_{S_1} E \cdot dS$ and $\iint_{S_2} \bar{E} \cdot d\bar{S}$, where S_1 is the hemisphere surface, S_2 is the flat disk in the xy-plane.

A normal vector (outward) to S_1 is $2x\bar{i} + 2y\bar{j} + 2z\bar{k}$ and a unit normal vector is $\bar{n} = x\bar{i} + y\bar{j} + z\bar{k}$.

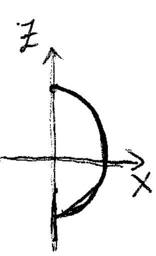
So
$$\iint_{S_1} E \cdot d\bar{S} = \iint_{S_1} E \cdot \bar{n}\, dS = \iint_{S_1} \underbrace{(2x^2+2y^2+2z^2)}_{=2 \text{ on } S_1} dS$$
$$= 2 \text{ area}(S_1) = 2 \cdot \frac{4\pi}{2} = 4\pi.$$

A normal unit vector outward to S_2 is $\bar{n} = -\bar{k}$

$$\iint_{S_2} \bar{E} \cdot d\bar{S} = \iint_{S_2} E \cdot \bar{n}\, dS = \iint_{S_2} \underbrace{(-2z)}_{=0 \text{ on } S_2} dS_2 = 0.$$

So $\iint_S \bar{E} \cdot d\bar{S} = \boxed{4\pi}$

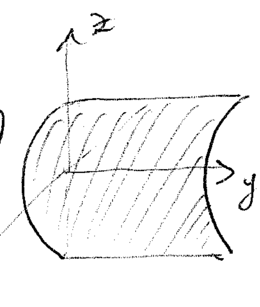
4/7.6 We need to use cylindrical coordinates (the trick from #1 is not really working)



$x = \cos\theta$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ (half cylinder)

$y = y$ $\quad 0 \le y \le 1$

$z = \sin\theta$



$$T_y \times T_\theta = \cos\theta\, \bar{i} + \sin\theta\, \bar{k} \text{ (points outward)}$$

$$\iint_S F \cdot d\bar{S} = \int_{y=0}^{1} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{y}\, \bar{i} \cdot (\cos\theta\, \bar{i} + \sin\theta\, \bar{k})\, d\theta\, dy$$
$$= \int_0^1 \int_{-\pi/2}^{\pi/2} \sqrt{y} \cdot \cos\theta\, d\theta\, dy = \frac{4}{3}.$$

9/7.6 A unit normal to the unit sphere is

$$x\bar{i} + y\bar{j} + z\bar{k}$$

$$\iint_S \bar{V}\cdot d\bar{S} = \iint_S \bar{V}\cdot\bar{n}\, dS = \iint_S (3x^2y^2 + 3x^2y^2 + z^4)\, dS$$

$$= \iint_S (6x^2y^2 + z^4)\, dS.$$

We need to parametrize S using spherical coord:

$\|T_\varphi \times T_\theta\| = \sin\varphi$ and

$$\iint_S (6x^2y^2 + z^4)\, dS = \int_0^\pi \int_0^{2\pi} (6\cos^2\theta \sin^2\theta \sin^4\varphi + \cos^4\varphi)\cdot \sin\varphi\, d\theta\, d\varphi$$

$$= \int_0^\pi \int_0^{2\pi} 6\left(\frac{\sin 2\theta}{2}\right)^2 \cdot \sin^5\varphi + \cos^4\varphi \sin\varphi\, d\theta\, d\varphi$$

$$= \int_0^\pi \int_0^{2\pi} \frac{3}{2}\sin^5\varphi \cdot \frac{1-\cos 4\theta}{2} + \cos^4\varphi \sin\varphi\, d\theta\, d\varphi$$

$$= \int_0^\pi \frac{3}{2}\sin^5\varphi \left(\frac{1}{2}\theta - \frac{\sin 4\theta}{8}\right)\Big|_0^{2\pi} + 2\pi\cdot \cos^4\varphi \sin\varphi\, d\varphi$$

$$= \int_0^\pi \left(\frac{3}{2}\sin^5\varphi \cdot \pi + 2\pi \cos^4\varphi \sin\varphi\right) d\varphi$$

$$\overset{\text{formula}}{\underset{\#18}{=}} \frac{3}{2}\pi\left(-\frac{\sin^4\varphi\cos\varphi}{5} + \frac{4}{5}\left(\frac{\sin^3\varphi\cos\varphi}{3} - \frac{2}{3}\cos\varphi\right)\right)\Big|_0^\pi$$

$$- 2\pi\cdot \frac{\cos^5\varphi}{5}\Big|_0^\pi = \frac{\cancel{3}}{\cancel{2}}\pi\cdot\frac{4}{5}\left(-\frac{2}{\cancel{3}}\right)\cdot(-\cancel{2}) + 2\pi\cdot\frac{2}{5}$$

$$= \frac{8}{5}\pi + \frac{4\pi}{5} = \boxed{\frac{12\pi}{5}}$$