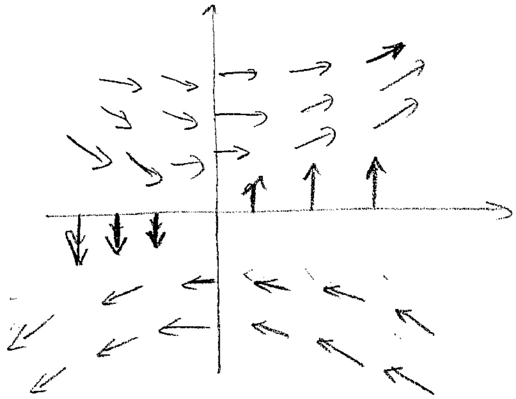
Math 212 - Hw 11-12

5/4.3



15/4.3

$\bar{c}'(t) = (\cos t, -\sin t, e^t)$

$\bar{F}(\bar{c}(t)) = (\cos t, -\sin t, e^t)$

So $\bar{F}(\bar{c}(t)) = \bar{c}'(t)$.

3/7.1 (a) $\int_{\bar{c}} f(x,y,z)\,ds = \int_0^1 f(\bar{c}(t))\,\|\bar{c}'(t)\|\,dt$

$= \int_0^1 e^t\sqrt{(2t)^2}\,dt = \int_0^1 2t\cdot e^t\,dt = \boxed{2}$

(b) $\int_{\bar{c}} f(x,y,z)\,ds = \int_1^3 6t^2\sqrt{1+9+4} = \boxed{52\sqrt{14}}$

5/7.1 $\|\bar{c}'(t)\| = \sqrt{\left(\frac{1}{t}\right)^2 + 1^2} = \sqrt{\frac{1}{t^2}+1}$

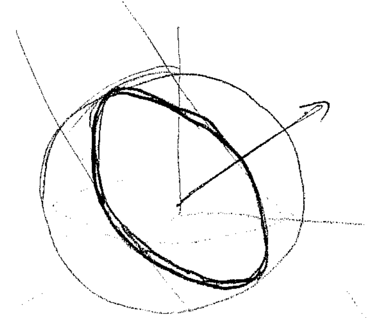
$\int_{\bar{c}} f(x,y,z)\,ds = \int_1^e t^{-3}\sqrt{1+t^{-2}} = -\frac{1}{3}\left(1+t^{-2}\right)^{\frac{3}{2}}\Big|_1^e$

$= \boxed{-\frac{1}{3}\left(1+\frac{1}{e^2}\right)^{\frac{3}{2}} + \frac{1}{3}\,2\sqrt{2}}$

14/7.1 $\int_{\bar{c}} f\,ds = \int_0^{t_0} t\sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1^2}$

$= \int_0^{t_0} t\sqrt{t^2+2} = \frac{1}{3}\left(t^2+2\right)^{\frac{3}{2}}\Big|_0^{t_0} = \boxed{\frac{1}{3}\left[\left(t_0^2+2\right)^{\frac{3}{2}} - 2\sqrt{2}\right]}$

13/7.1



This solution is a bit tricky because it uses a change of the coordinate system. Notice that the intersection of the sphere and the plane is a tilted circle. We pick a new orthogonal system such that the plane $x+y+z=0$ becomes horizontal. The normal to the plane i $\bar{i}+\bar{j}+\bar{k}$, hence a unit normal is $\frac{1}{\sqrt{3}}(\bar{i}+j+\bar{k})$. This is the new vertical axis.

For new horizontal axes pick two orthogonal vectors in the $x+y+z=0$. For example $(1,0,-1)$ & $(1,-2,1)$. Normalize them to $\bar{u}=\frac{1}{\sqrt{2}}(1,0,-1)$ $\bar{v}=\frac{1}{\sqrt{6}}(1,-2,1)$.

Under these new coordinates the circle is parametrized as: $\bar{c}(t) = (\cos\theta)\bar{u} + \sin\theta\bar{v}$.

$$\left((\cos\theta)\frac{1}{\sqrt{2}}(1,0,-1) + \sin\theta\frac{1}{\sqrt{6}}(1,-2,1)\right)$$

so $x = \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{6}}\sin\theta$ and

$$\int_C x^2\,ds = \int_0^{2\pi}\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{6}}\sin\theta\right)^2 \underbrace{|\bar{c}'(\theta)}_{=1}\,d\theta|$$

$$= \int_0^{2\pi}\left(\frac{1}{2}\cos^2\theta + \frac{2\sin\theta\cos\theta}{\sqrt{12}} + \frac{1}{6}\sin^2\theta\right)d\theta = \frac{2\pi}{3}.$$

1/7.2 c) $$\int_{\bar c} \bar F\cdot d\bar s = \int_0^{2\pi} \bar F(\bar c(t))\cdot \bar c\,'(t)\,dt$$

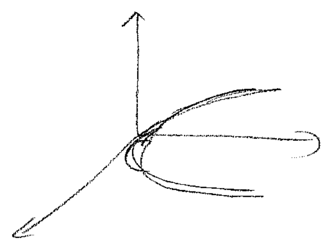
$$= \int_0^{2\pi} (\sin t\,\bar i + \cos t\,\bar k)\cdot(\cos t\,i - \sin t\,\bar k)\,dt$$

$$= \int_0^{2\pi} 0\,dt = \boxed{0}$$

d) $$\int_{\bar c} \bar F\cdot d\bar s = \int_{-1}^{2} (t^2\bar i + 3t\,\bar j + 2t^3\bar k)\cdot(2t\,\bar i + 3\bar j + 6t^2\bar k)\,dt$$

$$= \int_{-1}^{2} (12t^5 + 2t^3 + 9t)\,dt = \boxed{147}$$

3/7.2 Parametrization of $\bar c$:



$\bar c(t) = (t, t^2, 0) \quad -1 \le t \le 2$

$$W = \int_{\bar c} \bar F\cdot d\bar s = \int_{-1}^{2} (t\,\bar i + t^2\bar j)\cdot(\bar i + 2t\,\bar j)\,dt$$

$$= \int_{-1}^{2} (t + 2t^3)\,dt = \boxed{9.}$$

9/7.2 $$\int_{\bar c} \bar F\cdot d\bar s = \int_0^{2\pi} F(\bar c(t))\cdot \bar c\,'(t)\,dt$$

$$= \int_0^{2\pi} (\cos^3 t\,\bar i + \sin^3 t\,\bar j)\cdot(-3\cos^2 t\sin t\,\bar i + 3\sin^2 t\cos t\,\bar j)\,dt$$

$$= \int_0^{2\pi} (-3\cos^5 t\sin t + 3\sin^5 t\cos t)\,dt$$

$$= \frac{\cos^6 t}{2}\Big|_0^{2\pi} + \frac{\sin^6 t}{2}\Big|_0^{2\pi} = \boxed{0.}$$

Remark. Notice that $\bar F = \nabla(xy)$ so $\int_{\bar c} \bar F\cdot d\bar s = 0$ because $\bar c$ is a closed curve.

15/7.2 $\displaystyle\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz = ?$

Notice that $\bar{F} = 2xyz\,\bar{i} + x^2z\,\bar{j} + x^2y\,\bar{k}$

$= \nabla(x^2yz)$ hence

$$\int_C \bar{F}\,d\bar{s} = x^2yz\Big|_{(1,1,1)}^{(1,2,4)} \boxed{=7.}$$

Remark: It is essential to notice that $\bar{F} = \nabla f$, b/c the curve is specified only by its end points.

1/7.3

$$T_u \times T_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 2u & 0 \\ 0 & 1 & 2v \end{vmatrix}$$

At $(0,1,1)$ $u=0$ and $v=1$ and

$$T_u \times T_v\Big|_{(0,1)} = -4\bar{j} + 2\bar{k}.$$

Tangent plane: $-4(y-1) + 2(z-1) = 0$

$\underline{-4y + 2z = -2}$ or $\underline{2y - z = 1}$

5/7.3. Notice that the surface is the unit sphere (u, v are the spherical coordinates).

$T_u \times T_v = (\cos v \sin^2 u)\,\bar{i} + (\sin v \sin^2 u)\,\bar{j} + (\cos u \sin u)\,\bar{k}.$

$\|T_u \times T_v\| = \sin u$ so a unit normal vector is

given by $\bar{n} = \dfrac{T_u \times T_v}{\|T_u \times T_v\|} \boxed{= (\cos v \sin u)\,\bar{i} + (\sin v \sin u)\,\bar{j} + \cos u\,\bar{k}}$

11. Since the surface is the graph of $z = 3x^2 + 8xy$ a natural parametrization is:

$$\begin{cases} x = u \\ y = v \\ z = 3u^2 + 8uv \end{cases}$$

$$T_u \times T_v = \left(-\frac{\partial z}{\partial u}\right)\bar{i} + \left(-\frac{\partial z}{\partial v}\right)\bar{j} + \bar{k}$$

$$= (-6u - 8v)\,\bar{i} + (-8u)\,\bar{j} + \bar{k}.$$

At $(1,0,3)$. $u = 1,\ v = 0;\ T_u \times T_v = -6\bar{i} - 8\bar{j} + \bar{k}$

Tangent plane: $-6(x-1) - 8(y-0) + (z-3) = 0$

$-6x - 8y + z = -3$ or

$$\boxed{6x + 8y - z = 3} \quad (*)$$

Using the formula from page 133 (textbook)

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

$$z = 3 + 6(x-1) + 8(y-0)$$

$$\boxed{z = 3 + 6x + 8y - 6 = 6x + 8y - 3}$$ which is the same as $(*)$.

14/7.3 (a) The point $(1,1,\sqrt{2})$ corresponds to $\begin{cases} \theta = \frac{\pi}{4} \\ \varphi = \frac{\pi}{4} \end{cases}$

Compute $T_\varphi \times T_\theta$ at $(\frac{\pi}{4}, \frac{\pi}{4})$ and get

$$\sqrt{2}\,\bar{i} + \sqrt{2}\,\bar{j} + 2\bar{k}.$$

Tangent plane: $\sqrt{2}(x-1) + \sqrt{2}(y-1) + 2(z-\sqrt{2}) = 0$

or $\boxed{x + y + \sqrt{2}\,z = 4.}$

(b) A normal vector is $2x\,\bar{i} + 2y\,\bar{j} + 2z\,\bar{k}$

At $(1,1,\sqrt{2})$ one gets: $2\,\bar{i} + 2\,\bar{j} + 2\sqrt{2}\,\bar{k}$.

The equation of the tangent plane will be

$$\boxed{x + y + \sqrt{2}\,z = 4}$$

(c) Tangent plane at $(1,1,\sqrt{2})$:

$$z = g(1,1) + \frac{\partial g}{\partial x}(1,1)(x-1) + \frac{\partial g}{\partial y}(1,1)(y-1)$$

$$= \sqrt{2} - \frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1)$$

$$z = \sqrt{2} - \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y + \sqrt{2} \quad \text{or}$$

$$\boxed{x + y + \sqrt{2}\,z = 4.}$$

<u>Remark</u>: The answer should be the same in each case.