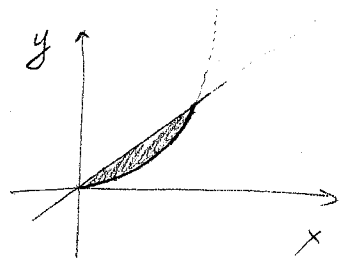
Hw #10 - Math 212

3/6.3



$$\bar{x} = \frac{\iint_D x\sigma(x,y)\,dx\,dy}{\iint_D \sigma(x,y)\,dx\,dy} \; ; \quad \bar{y} = \frac{\iint_D y\,\sigma(x,y)\,dx\,dy}{\iint_D \sigma(x,y)\,dx\,dy}$$

$D: \quad 0 \le x \le 1$

$\quad\;\; x^2 \le y \le x$.

$$\iint_D \sigma(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^{x} (x+y)\,dy\,dx = \frac{3}{20}$$

$$\iint_D x\sigma(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^{x} x(x+y)\,dy\,dx = \frac{11}{120}$$

$$\iint_D y\,\sigma(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^{x} y(x+y)\,dy\,dx = \frac{13}{168}$$

Hence $\bar{x} = \frac{11}{120} \cdot \frac{20}{3} = \underline{\underline{\frac{11}{18}}}$; $\bar{y} = \frac{13}{168} \cdot \frac{20}{3} = \underline{\underline{\frac{65}{126}}}$

10/6.3 $\quad \bar{x} = \dfrac{\iiint_D x\sigma(x,y,z)\,dx\,dy\,dz}{\iiint_D \sigma(x,y,z)\,dx\,dy\,dz}$; $\bar{y} = \dots$, $\bar{z} = \dots$

We use cylindrical coordinates to compute the integrals:

$$\iiint_D \sigma(x,y,z)\,dx\,dy\,dz = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \int_{z=1}^{2} r^2 z^2\, r\,dz\,d\theta\,dr = \frac{7\pi}{6}$$

$$\int_{r=0}^{1}\int_{0}^{2\pi}\int_{1}^{2} r\cos\theta\,(r^3z^2)\,dz\,d\theta\,dr = \int_0^1\int_0^{2\pi} r^4\cos\theta\cdot\left(\frac{8}{3}-\frac{1}{3}\right)d\theta\,d$$

$$= \frac{7}{3}\cdot 0 = 0. \Rightarrow \bar{x} = 0.$$

Similarly $\int_0^1\int_0^{2\pi}\int_1^2 r\sin\theta(r^3z^2)\,dz\,d\theta\,dr = 0 \Rightarrow \bar{y} = 0.$

$$\int_0^1\int_0^{2\pi}\int_1^2 r^3z^3\,dz\,d\theta\,dr = 2\pi\frac{15}{4}\int_0^1 r^3\,dr = \frac{15\pi}{2}\cdot\frac{1}{4} = \frac{15\pi}{8}$$

Hence $\bar{z} = \frac{15\pi}{8}\Big/\frac{7\pi}{6} = \frac{45}{28}.$

The center of mass is $\left(0, 0, \frac{45}{28}\right).$

1/6.4 $\int_0^1\int_0^1 \frac{1}{\sqrt{xy}}\,dx\,dy \overset{\text{Fubini}}{=} \int_0^1\left(\int_0^1 \frac{1}{\sqrt{xy}}\,dx\right)dy$

$$= \int_0^1 \frac{1}{\sqrt{y}}\cdot 2\sqrt{x}\Big|_{x=0}^{1} = \int_0^1 \frac{2}{\sqrt{y}}\,dy = 4\sqrt{y}\Big|_0^1 = 4.$$

One needs to think of these integrals as being improper at 0

16/6.4 Use spherical coordinates:

$$\int_{\rho=1}^{\infty}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} \frac{1}{\rho^4}\rho^2\sin\varphi\,d\varphi\,d\theta\,d\rho =$$

$$= 2\pi\int_{\rho=1}^{\infty} \rho^{-2}\cdot(-2)\,d\rho = 4\pi\cdot\frac{1}{\rho}\Big|_1^{\infty} = 4\pi$$

3/4.1 $\bar{v}(t) = \sqrt{2}\,\bar{i} + e^{t}\bar{j} - e^{-t}\bar{k}$; $\bar{a}(t) = 0\cdot\bar{i} + e^{t}\bar{j} + e^{-t}\bar{k}$

$\bar{v}(0) = \sqrt{2}\,\bar{i} + \bar{j} - \bar{k}$ $\qquad$ $\bar{a}(0) = \bar{j} + \bar{k}$.

Tangent line at $(0,1,1)$: $\bar{\ell}(t) = r(0) + t\,\bar{v}(0)$

$$\boxed{= t\sqrt{2}\,\bar{i} + (1+t)\bar{j} + (1-t)\bar{k}.}$$

7/4.1 $\frac{d}{dt}\left[\bar{c}_1(t)\times\bar{c}_2(t)\right] = \bar{c}_1'(t)\times\bar{c}_2(t) + \bar{c}_1(t)\times\bar{c}_2'(t)$

$$RHS = \begin{bmatrix} i & j & k \\ e^{t} & \cos t & 3t^2 \\ e^{-t} & \cos t & -2t^3 \end{bmatrix} + \begin{bmatrix} i & j & k \\ e^{t} & \sin t & t^3 \\ -e^{-t} & -\sin t & -6t^2 \end{bmatrix}$$

$$= (-2t^3\cos t - 3t^2\cos t)\bar{i} - (-2t^3e^{t} - 3t^2e^{-t})\bar{j} + (e^{t}\cos t - e^{-t}\cos t)\bar{k}$$

$$+ (-6t^2\sin t + t^3\sin t)\bar{i} - (-e^{t}6t^2 + t^3 e^{-t})\bar{j} + (-e^{t}\sin t + e^{-t}\sin t)\bar{k}$$

$$LHS = \frac{d}{dt}\begin{bmatrix} \bar{i} & \bar{j} & \bar{k} \\ e^{t} & \sin t & t^3 \\ e^{-t} & \cos t & -2t^3 \end{bmatrix} = \frac{d}{dt}\Big[(-2t^3\sin t + t^3\cos t)\bar{i} - (-e^{t}2t^3 - t^3e^{-t})\bar{j} + (e^{t}\cos t - e^{-t}\sin t)k\Big]$$

The two formulas are equal.

17/4.1 $\bar{c}(t) = \int c'(t)\,dt = \left(\frac{t^2}{2} + c_1\right)\bar{i} + \left(e^{t} + c_2\right)\bar{j} + \left(\frac{t^3}{3} + c_3\right)\bar{k}$

Since $\bar{c}(0) = (0,-5,1) \Rightarrow \boxed{\bar{c}(t) = \frac{t^2}{2}i + (e^{t} - 6)\bar{j} + \left(\frac{t^3}{3} + 1\right)\bar{k}}$

5/4.2 $L(\bar{c}) = \int_1^2 \|\bar{c}'(t)\| dt = \int_1^2 \sqrt{1^2 + 1^2 + (2t)^2}\, dt$

$= \int_1^2 \sqrt{2 + 4t^2}\, dt = 2\int_1^2 \sqrt{t^2 + \frac{1}{2}}\, dt$

$= 2\left(\frac{t}{2}\sqrt{t^2+\frac{1}{2}} + \frac{1}{4}\log\left|t + \sqrt{t^2 + 1/2}\right)\right|_1^2$

$= \underline{2\left(\sqrt{9/2} + \frac{1}{4}\log\left(2 + \sqrt{9/2}\right) - \frac{1}{2}\frac{\sqrt{3}}{2} - \frac{1}{4}\log\left(1 + \sqrt{\frac{3}{2}}\right)\right)}$

$= \boxed{\frac{6-\sqrt{3}}{2} + \frac{1}{2}\log\frac{2\sqrt{2}+3}{\sqrt{2}+\sqrt{3}}}$

7/4.2 $L(\bar{c}) = \int_0^{2\pi} \|\bar{c}_1'(t)\| dt + \int_{2\pi}^{4\pi} \|\bar{c}_2'(t)\| dt$

$= \int_0^{2\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 1^2}\, dt + \int_{2\pi}^{4\pi} \sqrt{1^2 + 1^2}\, dt$

$= \int_0^{2\pi} \sqrt{5}\, dt + \int_{2\pi}^{4\pi} \sqrt{2}\, dt = \boxed{2\pi(\sqrt{5} + \sqrt{2})}$

9/4.2 The points $(2,1,0)$ & $(4,4,\log 2)$ correspond to $t=1$, $t=2$.

$L(\bar{c}(t)) = \int_1^2 \sqrt{4 + (2t)^2 + (\frac{1}{t})^2}\, dt = \int_1^2 \sqrt{(2t + \frac{1}{t})^2}\, dt$

$= \int_1^2 2t + \frac{1}{t}\, dt = t^2 + \log t \Big|_1^2 = \boxed{3 + \log 2}$