MATH 435 - Homework #4 (Due 10/14)

1. Discuss the phase portrait of the system

$$\begin{aligned}\dot{x} &= -y + x(x^2+y^2-1)^2 \\ \dot{y} &= x + y(x^2+y^2-1)^2 .\end{aligned}$$

   Hint: Rewrite the system in polar coordinates.

2. Consider the system

$$\begin{aligned}\dot{x} &= a + x^2 - xy \\ \dot{y} &= y^2 - x^2 - 1 ,\end{aligned}$$

   where a is a parameter.

   (a) Sketch the phase portrait for $a = 0$. Show that there is a trajectory connecting two saddle points. (Such a trajectory is called a *saddle connection.*)

   (b) With the aid of `pplane` if necessary, sketch the phase portrait for $a < 0$ and $a > 0$.

   Notice that for $a \neq 0$, the phase portrait has a different topological character: the saddles are no longer connected by a trajectory. The point of this exercise is that the phase portrait in (a) is not structurally stable, since its topology can be changed by an arbitrary small perturbation of a.

3. Find the first three successive approximations $\mathbf{u}^1(t,\mathbf{y})$, $\mathbf{u}^2(t,\mathbf{y})$ $\mathbf{u}^3(t,\mathbf{y})$ of $W^s(\mathbf{0})$ for

$$\begin{aligned}\dot{x}_1 &= -x_1 \\ \dot{x}_2 &= x_2 + x_1^2\end{aligned}$$

   Note that $\mathbf{u}^3(t,\mathbf{y}) = \mathbf{u}^2(t,\mathbf{y})$ and so the sequence $\mathbf{u}^j(t,\mathbf{y})$ stabilizes at $\mathbf{u}^2(t,\mathbf{y})$ which gives the exact function defining locally $W^s(\mathbf{0})$. Show also that the unstable manifold coincides with the vertical axis $x_1 = 0$. Actually the formulas obtained determine the global stable and unstable manifolds. Prove this by solving explicitly the system.

   Recall that the sequence $\mathbf{u}^j(t,\mathbf{y})$ is constructed inductively from $\mathbf{u}^0(t,\mathbf{y}) = 0$ and

$$\mathbf{u}^{j+1}(t,\mathbf{y}) = U(t)\mathbf{y} + \int_0^t U(t-s)\mathbf{g}(\mathbf{u}^j(s,\mathbf{y}))ds - \int_t^\infty V(t-s)\mathbf{g}(\mathbf{u}^j(s,\mathbf{y}))ds .$$

   The matrices $U(t)$, $V(t)$ were defined in class. The vector $\mathbf{y}$ can be considered to have the last $n-k$ components 0 (where k is the dimension of the stable subspace). Then the function $\sigma^s = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ is defined as

$$\sigma_i(y_1,\ldots,y_k) = u_{k+i}(0, y_1, y_2, \ldots, y_k, 0, 0, \ldots, 0) \;, i = 1, \ldots k.$$

   For two dimensional systems (with one-dimensional stable manifold), the formulas are a bit easier: there is one real-function $\sigma^s(y_1) = u_2(0; y_1, 0)$, and the local stable manifold is given by the graph of σ^s, i.e. the pair $(y_1, \sigma^s(y_1))$ for y_1 in a small interval of E^s.

4. Consider the system $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1 - \cos(\theta)$, where r, θ represent polar coordinates. Sketch the phase portrait and show that the fixed point $r^* = 1$, $\theta^* = 0$ is weakly asymptotically stable but not Lyapunov stable.