## The Exponential of a Matrix

The exponential of the $n \times n$ matrix A is the $n \times n$ matrix

$$e^A = I + A + \frac{1}{2!}\,A^2 + \frac{1}{3!}\,A^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}\,A^k\,.$$

**Properties**

- The solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \mathbf{x}(0) = \mathbf{x}_0$$

  is given by $\mathbf{x}(t) = e^{tA}\mathbf{v}$, where $e^{tA}$ is the exponential of the $t$-dependent matrix $tA$.
- If $\lambda$ is an eigenvalue of $A$ and $\mathbf{v}$ an eigenvector (i.e. $(A - \lambda I)\mathbf{v} = 0$), then

$$e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}\,.$$

**Generalized eigenvectors.** We say that a vector $\mathbf{v}$ is a generalized eigenvector for the eigenvalue $\lambda$ if there exists a number $k > 1$ such that $(A - \lambda I)^k\mathbf{v} = 0$. One can show that if $\mathbf{v}$ is a generalized eigenvector, then

$$e^{tA}\mathbf{v} = e^{\lambda t}\left(\mathbf{v} + t(A-\lambda I)\mathbf{v} + \frac{t^2}{2!}(A-\lambda I)^2\mathbf{v} + \cdots + \frac{t^{k-1}}{(k-1)!}(A-\lambda I)^{k-1}\mathbf{v}\right)\,.$$

These properties help us find sufficient fundamental solutions for the linear system $\mathbf{x}' = A\mathbf{x}$. Recall that in the case of a repeated eigenvalue (of algebraic multiplicity $m$) we might not have $m$ linearly independent eigenvectors. This happens when the dimension of the nullspace of $A - \lambda I$ (called the geometric multiplicity of $\lambda$) is strictly less than the arithmetic multiplicity $m$. However, the mathematical theory says that one can complete the necessary number with linearly independent generalized eigenvectors. More precisely, there exists a power $k \leq m$, such that the nullspace of $(A - \lambda I)^k$ has dimension $m$, hence one can find $m$ linearly independent generalized eigenvectors. These generalized eigenvectors will provide the necessary fundamental solutions for our linear system.

### Procedure for $\lambda$ an eigenvalue of algebraic multiplicity $m$

To find $m$ linearly independent solutions associated with $\lambda$:

- Find the smallest integer $k \leq m$ such that null$(A - \lambda I)^k$ has dimension $m$.
- Find a basis $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_m$ of null$(A - \lambda I)^k$.
- For $j = 1, 2, \ldots, m$ compute the fundamental solutions

$$\mathbf{x}_j(t) = e^{tA}\mathbf{v}_j = e^{\lambda t}\left(\mathbf{v}_j + t(A-\lambda I)\mathbf{v}_j + \frac{t^2}{2!}(A-\lambda I)^2\mathbf{v}_j + \cdots + \frac{t^{k-1}}{(k-1)!}(A-\lambda I)^{k-1}\mathbf{v}_j\right)$$

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**Example.**

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{bmatrix}$$

*The characteristic polynomial* is

$$p(\lambda) = det(A - \lambda I) = -(\lambda + 3)(\lambda + 1)^2 \,,$$

hence the eigenvalues are $\lambda = -3$ and $\lambda = -1$ (with algebraic multiplicity 2).

For $\lambda = 3$: we find a corresponding eigenvector by solving $(A - 3I)\mathbf{v} = 0$. One gets, for example, $\mathbf{v}_1 = [-1, 3, 2]^T$. This gives us one fundamental solution

$$\mathbf{x}_1(t) = e^{3t} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} .$$

For $\lambda = 1$: solving for $(A - I)\mathbf{v} = 0$, we obtain a one-dimensional eigenspace spanned by $\mathbf{v}_2 = [-1, 2, 2]^T$. Hence a fundamental solution is given by

$$\mathbf{x}_2(t) = e^{t} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} .$$

We need a third solution. For this, we compute the nullspace of $(A - \lambda I)^2$ and find that it has dimension 2. Since the eigenvector $\mathbf{v}_2$ is also in the nullspace of $(A - \lambda I)^2$, we must choose a vector $\mathbf{v}_3$ in null$(A - \lambda I)^2$ which is not a multiple of $\mathbf{v}_2$. For example $\mathbf{v}_3 = [1, 0, 0]^T$, and this gives us a third solution

$$\mathbf{x}_3(t) = e^{t} \left(\mathbf{v}_3 + t(A + I)\mathbf{v}_3\right) = e^{t} \begin{bmatrix} 1 + 2t \\ -4t \\ -4t \end{bmatrix} .$$

In conclusion the general solution of $\mathbf{x}' = A\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t) \,.$$