Equithreshold Strata of Deformations of Plane Curve Singularities

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Problem

Find an improved method to calculate the log canonical threshold of a singular plane curve.

What Are Plane Curves?

A **plane curve** is a set X of zeros of a polynomial $f(x, y)$ with complex coefficients,

$$X = \{(x, y) | f(x, y) = 0\} \subset \mathbb{C}^2.$$

A plane curve f is **reduced** if and only if there is no polynomial g such that $g^2 | f$.

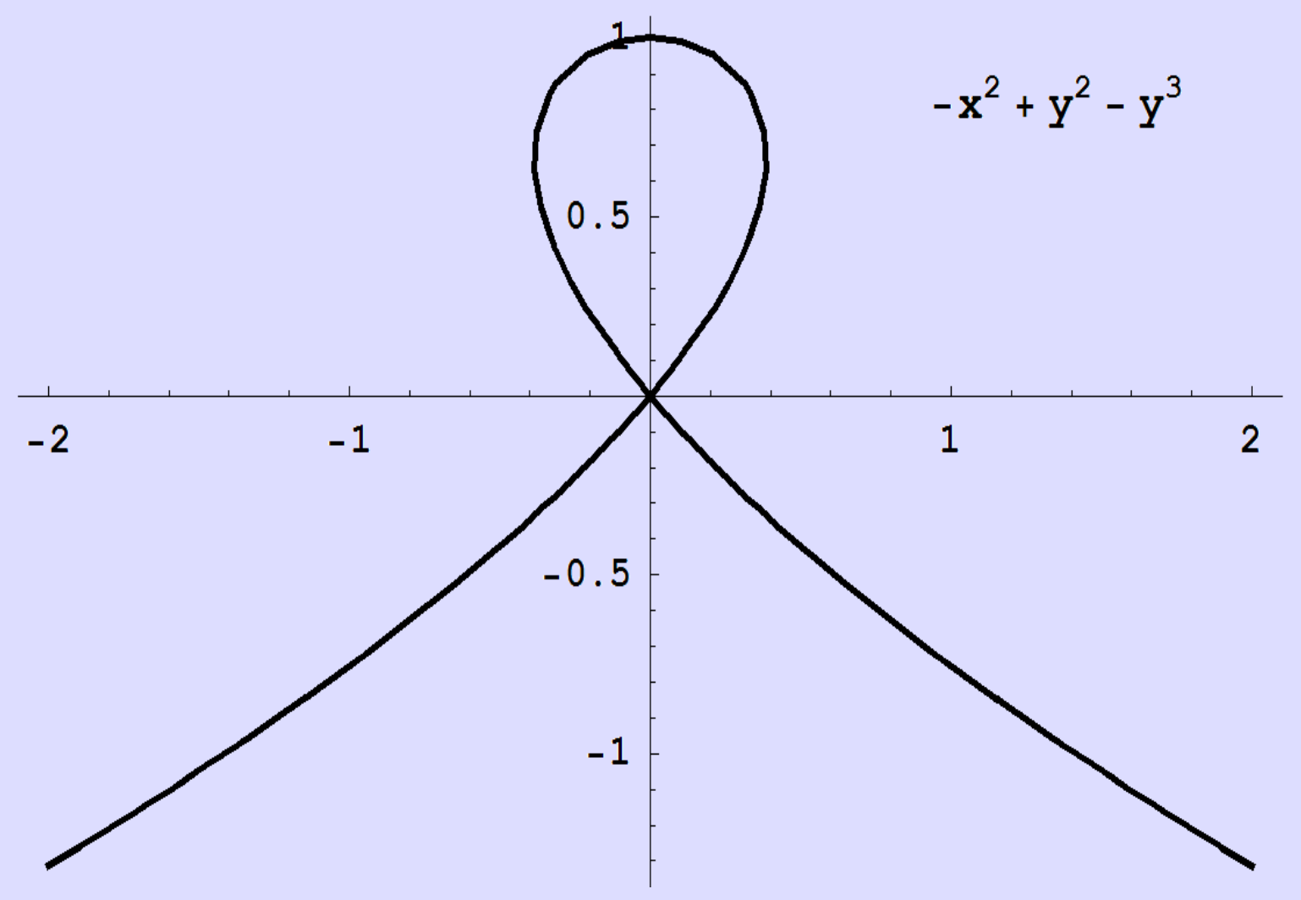
Even though a plane curve is a two-dimensional surface in a four-dimensional space ($\mathbb{C}^2$), we will still plot the curve in the real plane for demonstration purposes.

A **singularity** of a plane curve defined by $f(x, y) = 0$ is a point $(x_0, y_0) \in \mathbb{C}^2$ such that

$$f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0.$$

The plane curve is said to be **smooth** if and only if it has no singularities. Here is an example of the plane curve defined by

$$f(x, y) = x^2 - y^2 - y^3$$



Notice the singularity at the origin.

What Are Resolutions?

Smooth curves are much more readily understood than singular curves, so we analyze a singular curve X by realizing it as a projection of a smooth curve Y lying in a higher dimensional space. We call such a realization a **resolution.** In particular, a resolution is a (proper and birational) mapping $\pi : Y \to X$ of the form

$$\pi(x_1, \ldots, x_n) = \tfrac{p(x_1,\ldots,x_n)}{q(x_1,\ldots,x_n)}, \quad p, q \text{ polynomials}$$
$$\pi(x_1, \ldots, x_n) \in X, \quad \text{for all } (x_1, \ldots, x_n) \in Y.$$

Blowing Up

We can obtain a resolution using a technique called **blowing up**. The easiest way to understand this technique is through an example. Consider the plane curve X given by $f(x, y) = y^2 - x^2 - x^3$.

1. Find the singularities of X by solving

$$f = \tfrac{\partial f}{\partial x} = \tfrac{\partial f}{\partial y} = 0$$
$$y^2 - x^2 - x^3 = -2x - 3x^2 = 2y = 0,$$

   which gives $(x, y) = (0, 0)$ as its only solution.

2. For each singularity, use a substitution to shift the curve so the given singularity is at the origin. For our example, the only singularity is already at the origin.

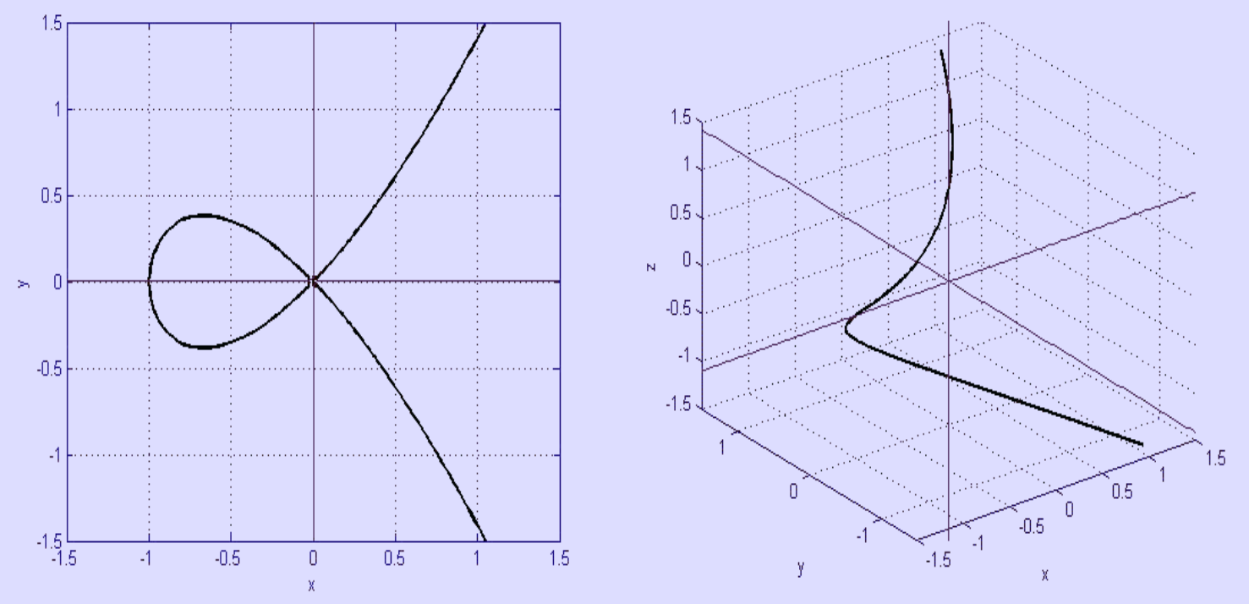
3. Perform another substitution that introduces a new dimension parameterized by z. We would like to stretch out the curve in the higher dimension, so we map each point $(x, y) \in X$ to the point $(x, y, x/y) \in Y$. To see if the new curve Y is now smooth, we look along the y-axis, i.e. we project onto the xz-plane by performing the substitution $y = zx$. The equation in the xz-plane is then,

$$y^2 - x^2 - x^3 \longrightarrow x^2(z^2 - 1 - x)\ .$$

   Note that this substitution introduced a spurious x^2 factor that would make the higher dimensional curve singular. Since we would like a smooth curve, we divide the transformed curve into two distinct parts: an exceptional curve and a proper transform. The **exceptional curve** E_1 is the component of the higher dimensional curve that maps onto the singularity (i.e. $x^2 = 0$). The **proper transform** $\tilde{X}$ is the closure of the remaining points ($z^2 - 1 - x = 0$).

   We could have just as easily projected onto the yz-plane and done a similar analysis. In general, all possible projections must be examined in order to discover any singularities remaining in the proper transform.

4. If the projections of the proper transform of the new curve Y onto the xz and yz-planes are smooth, then we are done. Elsewise, repeat the process.



As you can see, the blowup process transformed the singular plane curve on the left to a smooth curve in higher dimensional space, as seen on the right.

In this case, a resolution of the curve was obtained after a single blowing up, however this is not the case in general.

What is the Log Canonical Threshold?

In order to analyze the singularities of curves embedded in surfaces, we introduce a measure of complexity that takes into account both the nature of the surface and the embedded curve. As such, the log canincal threshold of a surface S and a curve X at a point P can be defined in terms of a log resolution π by

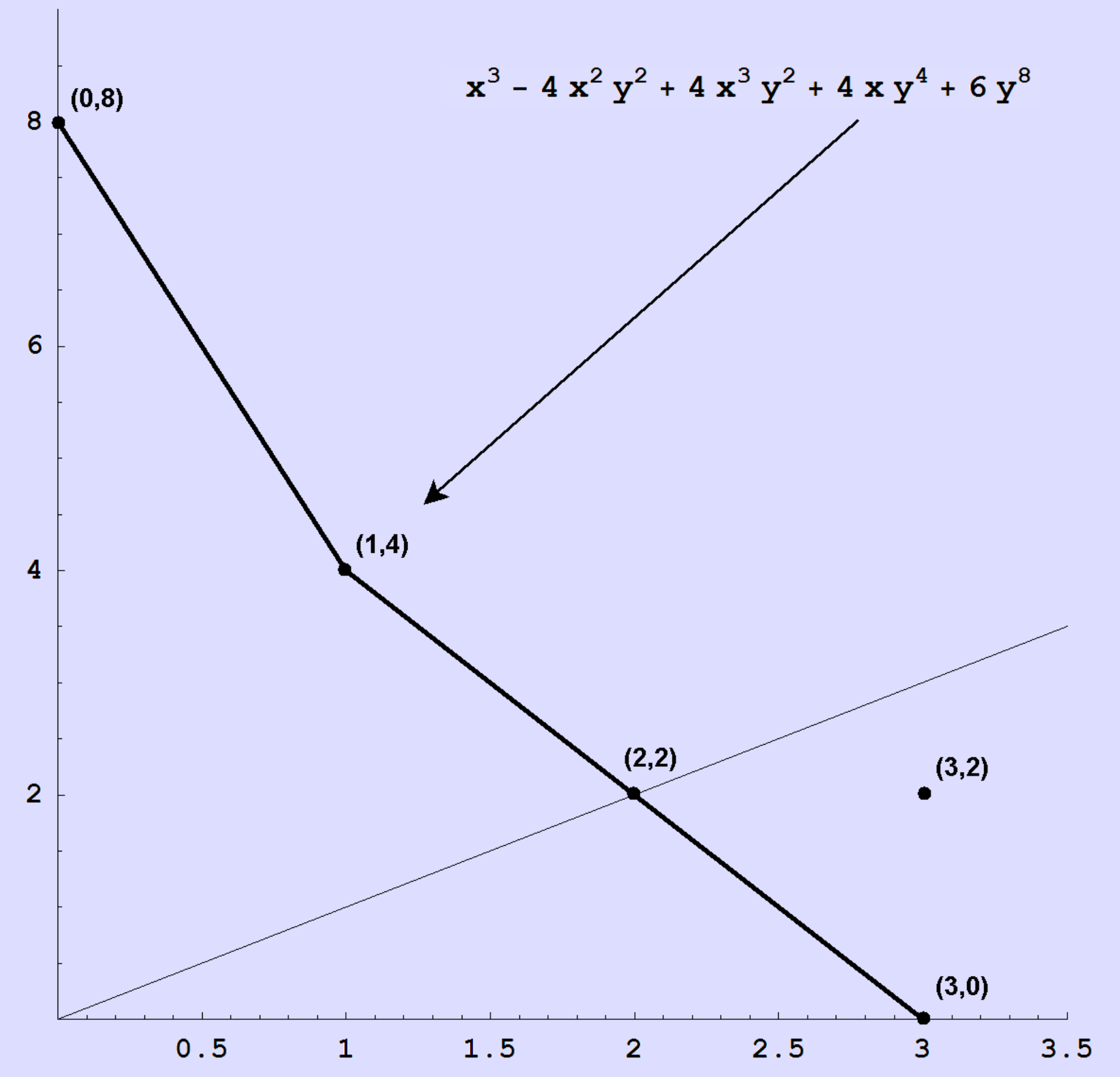
$$c_P(S, X) = \min_i \left\{ \frac{a_i + 1}{c_i} \middle| a_i = E_i.\pi^*(K_X), \quad c_i = E_i.\tilde{X} \right\}$$

Here a_i is the measure of how singular the surface is. Likewise, c_i is the measure of how singular the curve is. In practice, these values are obtained by blowing up curves and examining the intersection multiplicity of the exceptional curves and proper transform.

While the log caninical threshold is defined for more complicated surfaces, we only consider curves in the surface $\mathbb{C}^2$ because we are interested in plane curves. If the curve X is given by the polynomial f with singularity at $O = (0, 0)$, then we use the shorthand $c_O(f) = c_O(\mathbb{C}^2, X)$.

The Newton Polygon

Since we need a log resolution to calculate $c_O(f)$ using the above definition, it is difficult to compute. Our research sought to find better ways to calculate $c_O(f)$. The Newton Polygon provides a tool for achieving this. Given a polynomial $f(x, y) = \sum_i c_i x^{\alpha_i} y^{\beta_i}$, the **Newton Polygon** of f is the convex hull of the upper right quadrants of the points (α_i, β_i). The figure below shows the Newton Polygon of $f(x, y) = x^3 + 4x^3y^2 - 4x^2y^2 + 4xy^4 + 6y^8$.



In other words, we connect the "lower left" points corresponding to monomials in our polynomial and disregard any (non-zero) coefficients.

Finding $c_O(f)$ Using the Newton Polygon

Using a theorem proved by Kollár [1], we can find $c_O(f)$ for a large class of f using the Newton Polygon. Let $w = (a, b)$ be the normal vector of any edge of the Newton Polygon of f intsersecting the line $y = x$. We can organize the monomials of f by weighted degree, i.e. the weighted degree of $x^\alpha y^\beta$ is $w(x^\alpha, y^\beta) = a\alpha + b\beta$. The polynomial of terms of f of lowest weighted degree is written f_w. For example,

$$f(x, y) = \underbrace{x^3}_{6} + \underbrace{4x^3y^2}_{8} - \underbrace{4x^2y^2}_{6} + \underbrace{4xy^4}_{6} + \underbrace{6y^8}_{8},$$

where $w = (2, 1)$ is the normal to the edge intersecting the line $y = x$, and the numbers under the monomials are their weights with respect to w. Then f_w is composed of the degree six terms.

We can always factor f_w in the form $f_w = x^c y^d \prod_i (x^b + A_i y^a)^{e_i}$, where the A_i are distinct. In our example,

$$f_w(x, y) = x^3 - 4x^2y^2 + 4xy^4 = x(x - 2y^2)^2.$$

Once in this form, taking $e_m = \max\{c, d, e_i\}$, we determine whether our Newton Polygon method will work by checking whether

$$cb + ad \textstyle\sum_i e_i + ad \leq e_m(a + b)$$
$$1 \cdot 1 + 2 \cdot 0 \cdot 2 + 2 \cdot 0 \leq 2 \cdot (2 + 1)$$

If and only if this bound holds, the curve is said to be **log canonical away from the origin**. More importantly, if this bound holds, then we can calculate $c_O(f)$ by finding the point (p_0, p_0) at which the $y = x$ line intersects the Newton Polygon and realizing $c_O(f) = 1/p_i$. We find in our example that the point of intersection is $(2, 2)$, and thus $c_O(f) = 1/2$. If the above inequality does not hold, then in many cases the curve can still be modified (while preserving the threshold) using substitutions of the form $x \to (x - A_i y)$ such that the bound will hold.

Concluding Remarks

In sum, you can see that the process of disovering the log canonical threshold of a plane curve singularity at the origin can be particularly difficult. While our research provides a means for calculating $c_O(f)$ far more quickly than blowing up, future research in this topic may lead to finding suitable curve preparation techniques so that our method works for any plane curve. Additionally, characterizing the equithreshold ideal of a curve using our technique is an area ripe for research.

References

[1] Kollár, J. Singularities of pairs. *Proc. Sympos. Pure Math.*, **62**, 221-287, (1997).