Plane Curve Singularities and the Log-Canonical Threshold of $x^p - y^q = 0$

Abraham Taicher and Thomas Murphy

Advisors: Dr. Brendan Hassett and Amanda Knecht

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2 Introduction

In the background section of this paper we present an overview of some of the theory and methods associated with calculating the log-canonical threshold of complex plane curve singularities. While we present enough theory to provide sufficient context, we focus mainly on examples and methods in hope that of making the material more approachable for undergraduates. Following the background material is a detailed proof for finding the log-canonical threshold of $x^p + y^q$ for any positive integers p and q. We first introduce some important algebra about the Euclidean algorithm in the Algebraic Groundwork section. We then apply this algebra to our algorithm for finding the log-canonical threshold.

3 General Background

Definition 1 *A* ***complex plane curve*** *is the set of points in the complex plane where a non-constant polynomial vanishes. Symbolically we can write a complex plane curve $Z(f)$ as*

$$Z(f) := \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\} \subset \mathbb{C}^2$$

For example, given the polynomial $f = x^2 + y^2$, the associated plane curve, $Z(f)$, is $x^2 + y^2 = 0$.

While we restrict our discussion to *complex* plane curves, plane curves can be taken over any field. It is "nice" to work over the complex plane because it is an algebraically closed field. For the remainder of the paper plane curve and complex plane curve are taken to be synonymous unless otherwise noted.

If a field is algebraically closed, then every polynomial taken over that field has a zero in the field. It is easy to see that $x^2 + 1 = 0$ has no solutions in the real numbers so that the real numbers are not algebraically closed.

Now that we have defined plane curves we can discuss plane curve singularities. Essentially a singularity is a point where the curve does not behave nicely. The curve might come to a point as in a cusp, or it might cross itself as in a node.

Definition 2 *A plane curve $Z(f)$ is* ***singular*** *at a point (x_0, y_0) if*

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0.$$

i.e. a curve is singular at a point if both of the partial derivatives vanish at that point. A curve is said to be ***non-singular*** *if it has no singular points.*

For example, the curve $x^2 + y^2 = 0$ is singular at the origin because $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 2y$ so that both partials vanish at the point (0,0).

We can find all singular points of a curve by simultaneously solving the system of equations given by:

$$\begin{cases} f(x,y) = 0 \\ \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial x} = 0 \end{cases}$$

For example, we can find all the singular points of the curve $f(x,y) = x^3 - y^2 - x^4 - y^4$. We first take partial derivatives and set them equal to zero.

$$\frac{\partial f}{\partial x} = 3x^2 - 4x^3 = 0$$

$$\frac{\partial f}{\partial y} = 2y + 4y^3 = 0.$$

$x = 0$ or $\frac{3}{4}$ and $y = 0$ or $\pm i\sqrt{(\frac{1}{2})}$ are solutions. Thus we have 6 points that satisfy the condition on the partial derivatives, and after substituting each of these points into the original function f only the point (0,0) satisfies $f(x,y) = 0$.

Next we discuss linear change of coordinates and the concept of multiplicity. Given a point (a,b) on a plane curve, we can translate that point to the origin through a linear change of variables. If we let $\tilde{x} = x - a$ and $\tilde{y} = y - a$ then we have effectively shifted the original curve so that the point (a,b) is at the origin. This is because

$$(x,y) = (a,b) \Leftrightarrow (\tilde{x}, \tilde{y}) = (0,0)$$

Let $Z(f)$ be a plane curve and p be some point on that curve. Translate the curve so that p goes to the origin and the polynomial f is transformed to the polynomial g. Note that g will be of the form $g = g_1 + g_2 + \ldots$ where g_1 contains all terms of degree 1, g_2 contains all terms of degree 2, etc. Also note that none of $g_1, g_2, g_3, \ldots$ is necessarily non-zero. We can always write $g = g_i + g_{i+1} + g_{i+2}$ where g_i is the first non-zero homogeneous part. (This does not imply that $g_{i+1}, g_{i+2}, \ldots$ are non-zero).

Definition 3 *The **multiplicity** $\mu_p(f)$ of a point p on a plane curve $Z(f)$ is the degree of the non-zero monomial of lowest degree after we shift the point to the origin. In other words, using the notation above, $\mu_p(f) = i$.*

An example should make this clear. Consider $Z(f) = y^2 - x^2 - x^3$. The lowest degree terms are of degree two. Thus the multiplicity at the origin for the curve $Z(f)$ is two.

Theorem 1 *A point (a, b) on a plane curve $Z(f)$ is non-singular if and only if $\mu_p(f) = 1$.*

Proof: $\Rightarrow$ First note that the partial derivative is unchanged when we shift our curve to $g(x, y) = f(x + a, y + b)$ since

$$\frac{\partial g}{\partial x}(x, y) = \frac{\partial f}{\partial x}(x + a, y + b)$$

$$\frac{\partial g}{\partial y}(x, y) = \frac{\partial f}{\partial y}(x + a, y + b).$$

This implies that $\frac{\partial g}{\partial x}(0, 0) = \frac{\partial f}{\partial x}(a, b)$ and $\frac{\partial g}{\partial y}(0, 0) = \frac{\partial f}{\partial y}(a, b)$. Since $Z(f)$ is non-singular at (a, b) it must be true that either $\frac{\partial f}{\partial x}(a, b)$ or $\frac{\partial f}{\partial y}(a, b)$ is non-zero so the same is true for $\frac{\partial g}{\partial x}(0, 0)$ and $\frac{\partial g}{\partial y}(0, 0)$. This implies that $g(x, y)$ must have some term of degree one because the partial derivatives of all terms of higher order vanish at the origin. Thus $u_p(f) = 1$.

$\Leftarrow$ Let $\mu_p(f) = 1$. This implies that after we change coordinates $f = c_1\tilde{x} + c_2\tilde{y} + c_3\tilde{x}^2 + \ldots$ with either c_1 or c_2 non-zero. Without loss of generality assume that c_1 is non-zero. This implies that $\frac{\partial f}{\partial \tilde{x}}(0, 0) = c_1 \neq 0$. Since the partial derivatives are unchanged by linear transformations, this implies that $\frac{\partial f}{\partial x}(p) = c_1 \neq 0$. Thus $Z(f)$ is non-singular at p.

3.1 Blowing up to remove singularities

In order to "resolve" such a singularity we use a process referred to as blowing up. To properly define this method we must first introduce the notion of projective space, $\mathbb{P}^1$.

Formally, $\mathbb{P}^1$ is the quotient of $\{(t, u) \mid (t, u) \neq (0, 0)\}$ under the equivalence relation $\sim$. $(t_1, u_1) \sim (t_2, u_2)$ if there exists $\lambda \neq 0$ such that $(t_2, u_2) = \lambda(t_1, u_1)$. We use $[t, u]$ to denote these equivalence classes, which can be thought of as possible slopes of lines through the origin in $\mathbb{C}^2$.

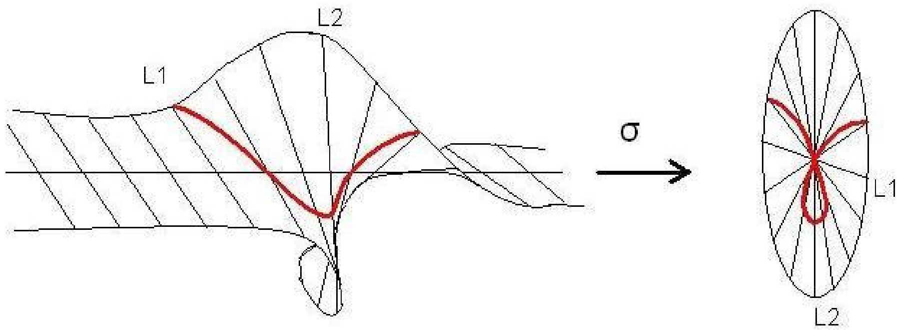
The blowup of $\mathbb{C}^2$ at the origin $\text{Bl}_{(0,0)}\mathbb{C}^2$ is the subvariety of $\mathbb{C}^2 \times \mathbb{P}^1$ defined by the equation $xu - yt = 0$.

$$\begin{array}{ccc} \mathrm{Bl}_{(0,0)}\mathbb{C}^2 & \subset & \mathbb{C}^2 \times \mathbb{P}^1 \\ & \overset{\sigma}{\searrow} & \downarrow p \\ & & \mathbb{C}^2 \end{array}$$

Note that the map p from $\mathbb{C}^2 \times \mathbb{P}^1$ to $\mathbb{C}^2$ is the projective map. For $(x_0, y_0) \neq (0,0)$,

$$\sigma^{-1}(x_0, y_0) = (x_0, y_0, [x_0, y_0])$$

which defines the inverse to σ over $\{\mathbb{C}^2 - (0,0)\}$. For $(x_0, y_0) = (0,0)$, we are solving $0t - 0u = 0$ so $\sigma^{-1}(0,0) = \mathbb{P}^1$. Geometrically, the proper transform of any lines with different slopes passing through $(0,0)$ will not intersect. With each successive blowup, the curve becomes simpler until the proper transform becomes nonsingular.(Brieskorn 462) The following picture depicts such a mapping:



If $t \neq 0$ then we can assume $t = 1$,

$$x(\frac{u}{t}) = y \quad \Rightarrow \quad xu = y$$

similarly if $u \neq 0$,

$$y(\frac{t}{u}) = x \quad \Rightarrow \quad yt = x$$

Consider a curve $Y \subset \mathbb{C}^2$ that has an isolated singularity at the origin.

To clarify, we look at an example. Let $Y \subset \mathbb{C}^2$ be the curve $y^2 = x^3$; Y is singular at the origin. We will blow up the curve at the origin O. Let t and u be the coordinates for $\mathbb{P}^1$.

We have the two equations $xu = yt$ and $y^2 = x^3$. $\mathbb{P}^1$ is covered by the open subsets $t \neq 0$ and $u \neq 0$; we will consider these as two separate cases. Assuming $t \neq 0$, we have the following equations:

$$y^2 = x^3 \qquad \text{and} \qquad y = xu$$

Substituting, we get that $x^2u^2 = x^3$, or $x^2(x - u^2) = 0$. The equation $x^2 = 0$ represents the *exceptional divisor* of the curve (denoted E_1), while $x - u^2 = 0$ represents the *proper transform* (denoted T_1). The exceptional divisor and the proper transform intersect at $u = 0$. We say that E_1 has multiplicity 2 i.e. $\mu_{(0,0)}(E_1) = 2$.

The computation for the case $u \neq 0$ is similar. Let $u = 1$, so that $x = yt$, and substitute in the equation for Y. The result is $y^2(1 + yt^3) = 0$. Here $E_1 \cap T_1 = \varnothing$.

Since the curve $x - u^2 = 0$ is nonsingular, we have resolved the singularity in Y at the point O. In some cases, especially with higher degrees, multiple blow-ups may be needed to resolve a singularity. Moreover, even when this is done, sometimes it is necessary to blow up again if the proper transform and the exceptional divisor do not have normal crossings. Normal crossing requires that the proper transform intersects normally with only one exceptional divisor.

Going back to our example, we see that $E_1 : x^2 = 0$ and $T_1 : x - u^2 = 0$ meet tangentially at the point $(0, 0)$. We resolve this problem by blowing up again.

In the second blow-up, we take $x^2(x - u^2) = 0$ to be a curve in $\mathbb{C}^2$, and introduce two new projective coordinates, say (a, b), for $\mathbb{P}^1$. These satisfy the condition $ax = bu$. Taking the open subsets $a \neq 0$ and $b \neq 0$ in $\mathbb{P}^1$, we get two new equations:

$$(1) \quad a \neq 0 \longrightarrow (bu)^2(bu - u^2) = 0 \longrightarrow b^2u^3(b - u) = 0$$

$$(2) \quad b \neq 0 \longrightarrow x^2(x - (ax)^2) = 0 \longrightarrow x^3(1 - a^2x) = 0$$

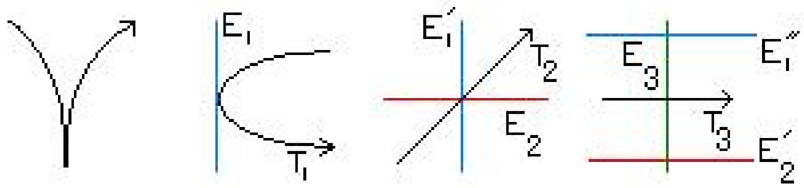
In these equations, we call the new proper transform and exceptional divisor T_2 and E_2, respectively. Note that, in (2), $T_2 \cap E_2 = \varnothing$. Thus, we only concern ourselves with (1). In this equation, E_2 is represented by $u^3 = 0$ with multiplicity 3,T_2 by $b - u = 0$, and $b^2 = 0$ represents the blow-up of E_1 (denoted E_1'). E_2 and T_2 intersect diagonally so we blow up again.

Starting with equation (1), we introduce the projective coordinates (r, s). Proceeding as before, we end up with two new equations:

$$(3) \quad s^3 b^6 (1 - s) = 0$$

$$(4) \quad r^2 u^6 (r - 1) = 0$$

These two curves are both nonsingular and have normal crossings. The intersection of E_3 and T_3 is $s = 1$ in (3) and $r = 1$ in (4). Thus, the blow-up computation is complete. The following visual shows the progression of the blowup process from left to right, beginning with a cusp and ending with normal crossings:



3.2 Calculating the Log-Canonical Threshold

Associated with each plane curve singularity is an important invariant α called the *log-canonical threshold.* This value can be determined by performing two calculations, one involving the multiplicity of the exceptional divisor at each stage of the blow-up and the other involving differential forms.

Let's consider again the curve $Y \subset \mathbb{C}^2$ defined by the equation $y^2 = x^3$. We are interested in the multiplicities of the exceptional divisors at each stage of the blowup process. The multiplicities of E_1, E_2, and E_3 are 2,3, and 6 respectively.

The next component of the log canonical threshold calculation involves a computation involving differential forms. For our purposes, all we need to know about the differential form $dx \wedge dy$ are two rules:

$$dx \wedge dx = 0$$

$$d(ux) = x\, du + u\, dx$$

In this computation, we start with $dx \wedge dy$ and then substitute in the same manner that we did in the original resolution of the singularity (e.g. substitute xu for y), counting multiplicities in the same fashion. We will label the first blow-up π_1 and the succeeding ones similarly:

$$\pi_1 * (dx \wedge dy) = dx \wedge d(ux) = dx \wedge (x\, du + u\, dx) = x\, dx \wedge du$$

The factor x signifies that there is a zero of multiplicity one along the first exceptional divisor E_1. Blowing up again, we get:

$$(\pi_2 \circ \pi_1) * (dx \wedge dy) = \pi_2 * (xdx \wedge du) = au[d(au) \wedge du] = au[uda \wedge du] = au^2 da \wedge du$$

Thus there is a zero of multiplicity 2 along E_2. After the final blowup:

$$(\pi_3 \circ \pi_2 \circ \pi_1) * (dx \wedge dy) = \pi_3 * (au^2 da \wedge du) = bu^3\, [u\, db \wedge du]$$

. So there is a zero of multiplicity 3 along E_3.

The log canonical threshold is

$$\alpha = min\{\frac{C_i + 1}{A_i}\} \qquad 1 \leq i \leq n$$

where C_i is equal to the multiplicity of the exceptional divisor after the i^{th} blowup in the progression of the differential forms. A_i is the multiplicity of the exceptional divisor after the i^{th} blowup. n is the total number of blowups. This yields three equations for $i = 1, 2, 3$:

$$min\{\frac{1+1}{2}, \frac{2+1}{3}, \frac{3+1}{6}\} = \frac{2}{3}$$

Thus the log canonical threshold for the curve $y^2 = x^3$ is $\frac{2}{3}$ at the origin.

4 Algebraic Groundwork

Let $P_n = P_n(k_0, k_1, \ldots, k_n)$ be a polynomial in $n+1$ variables defined recursively for all $n \geq 0$ with $P_{-2} = 0$ and $P_{-1} = 1$:

$$P_n(k_0, \ldots, k_n) = k_0 P_{n-1}(k_1, \ldots, k_n) + P_{n-2}(k_2, \ldots, k_n)$$

We introduce this polynomial to abstractly represent the Euclidean algorithm. This approach facilitates proving some useful formulas dealing with continued fractions.

Lemma 1

$$P_n(k_0, \ldots, k_n) = P_n(k_n, \ldots, k_0)$$

In other words, $P_n(k_0, \ldots, k_n)$ is unchanged if we replace $k_0, \ldots, k_n$ with $k_n, \ldots, k_0$

Proof:

$$\begin{cases} P_{-1} = 1, \\ P_0(k_0) = k_0, \\ P_1(k_0, k_1) = k_0 k_1 + 1 = P_1(k_1, k_0), \end{cases}$$

We can proceed by induction: assume,

$$P_m(k_{i_0}, k_{i_1}, \ldots, k_{i_m}) = P_m(k_{i_m}, k_{i_{m-1}}, \ldots, k_{i_0})$$

for $i_0, \ldots, i_m$ integers between 0 and m with $m < n$. Basically, any progression of k's can be reversed when $m < n$.

$$P_n(k_0, \ldots, k_n) = k_0 P_{n-1}(k_1, \ldots, k_n) + P_{n-2}(k_2, \ldots, k_n)$$

$$= k_0 P_{n-1}(k_n, \ldots, k_1) + P_{n-2}(k_2, \ldots, k_n)$$

$$= k_0[k_n P_{n-2}(k_{n-1}, \ldots, k_1) + P_{n-3}(k_{n-2}, \ldots, k_1)] + P_{n-2}(k_2, \ldots, k_n)$$

$$P_n(k_n, \ldots, k_0) = k_n P_{n-1}(k_{n-1}, \ldots, k_0) + P_{n-2}(k_{n-2}, \ldots, k_0)$$

$$= k_n P_{n-1}(k_0, \ldots, k_{n-1}) + P_{n-2}(k_0, \ldots, k_{n-2})$$

$$= k_n[k_0 P_{n-2}(k_1, \ldots, k_{n-1}) + P_{n-3}(k_2, \ldots, k_{n-1})] + P_{n-2}(k_0, \ldots, k_{n-2})$$

Canceling like terms,

$$P_n(k_0, \ldots, k_n) - P_n(k_n, \ldots, k_0)$$

$$= k_0 P_{n-3}(k_{n-2}, \ldots, k_1) + P_{n-2}(k_2, \ldots, k_n) - k_n P_{n-3}(k_2, \ldots, k_{n-1}) - P_{n-2}(k_0, \ldots, k_{n-2})$$

$$= P_{n-4}(k_2, \ldots, k_{n-2}) - P_{n-4}(k_2, \ldots, k_{n-2}) = 0$$

so $P_n(k_0, \ldots, k_n) = P_n(k_n, \ldots, k_0)$

q.e.d

Without loss of generality we set $P_n(k_0, k_1, \ldots, k_n) = P(k_0, k_1, \ldots, k_n)$ Next we fix a positive integer t (i.e. set $n = t$) and $t+1$ variables $k_0, \ldots, k_t$. Define a function $a(r, s)$ for any two integers (r,s) satisfying $0 \leq r \leq s \leq t$:

$$a(r, s) = \sum_{i=s-r}^{s} k_i P(k_{i+1}, \ldots, k_s) P(k_{i+1}, \ldots, k_t)$$

with $P(k_{s+1}, \ldots, k_s) = P_{-1} = 1$ and $P(k_{s+2}, \ldots, k_s) = P_{-2} = 0$. Note a(r,s) also depends on t.

Lemma 2

$$a(r,s) = \begin{cases} P(k_{s-r},\ldots,k_s)P(k_{s-r+1},\ldots,k_t), & r \text{ even;} \\ P(k_{s-r+1},\ldots,k_s)P(k_{s-r},\ldots,k_t), & r \text{ odd.} \end{cases}$$

Proof:

$$a(0,s) = k_s P(k_{s+1},\ldots,k_s)P(k_{s+1},\ldots,k_t)$$
$$= P(k_s)P(k_{s+1},\ldots,k_t)$$
$$a(1,s) = a(0,s) + k_{s-1}P(k_s)P(k_s,\ldots,k_t)$$
$$= P(k_s)P(k_{s+1},\ldots,k_t)+k_{s-1}P(k_s)P(k_s,\ldots,k_t) = P(k_s)[P(k_{s+1},\ldots,k_t)+k_{s-1}P(k_s,\ldots,k_t)]$$
$$= P(k_s)P(k_{s-1},\ldots,k_t)$$

This result holds true for $r = 0, 1$ so we can proceed by induction: assume r is even,

$$a(r-1,s) = P(k_{s-r+2},\ldots,k_s)P(k_{s-r+1},\ldots,k_t)$$
$$a(r,s) = a(r-1,s) + k_{s-r}P(k_{s-r+1},\ldots,k_s)P(k_{s-r+1},\ldots,k_t)$$
$$= P(k_{s-r+1},\ldots,k_t)[P(k_{s-r+2},\ldots,k_s) + k_{s-r}P(k_{s-r+1},\ldots,k_s)]$$
$$= P(k_{s-r+1},\ldots,k_s)P(k_{s-r},\ldots,k_t)$$

Next,

$$a(r+1,s) = P(k_{s-r},\ldots,k_s)[P(k_{s-r+1},\ldots,k_s) + k_{s-r-1}P(k_{s-r},\ldots,k_s)]$$
$$= P(k_{s-r},\ldots,k_s)P(k_{s-r+1},\ldots,k_t)$$

q.e.d

Let,

$$a(s,s) = a(s) = \begin{cases} P(k_0,\ldots,k_s)P(k_1,\ldots,k_t), & \text{s even;} \\ P(k_1,\ldots,k_s)P(k_0,\ldots,k_t), & \text{s odd.} \end{cases}$$

with $a(-2) = a(-1) = 0$.

Next define for $-2 \le s \le t$,

$$b(s) = a(s) + P(k_{s+2},\ldots,k_t)$$

Note that $a(t) = b(t)$ because $P(k_{t+2},\ldots,k_t) = 0$, but $a(t-1) \neq b(t-1)$
Finally define for $s \ge 0$

$$c(s) = \sum_{i=0}^{s} k_i P(k_{i+1},\ldots,k_s) + 1$$

with $c(-2) = c(-1) = 1$.

Lemma 3

$$c(s) = P(k_0, \dots, k_s) + P(k_1, \dots, k_s)$$

Proof:
expanding the sum,

$$c(s) = P(k_1, \dots, k_s) + k_1 P(k_2, \dots, k_s) + \dots + k_s P(k_{s+1}, \dots, k_s) + 1$$

$$= [P(k_0, \dots, k_s) - \cancel{P(k_2, \dots, k_s)}] + [P(k_1, \dots, k_s) - \cancel{P(k_3, \dots, k_s)}] + \dots + [\cancel{P(k_s)} - 1] + 1$$

after canceling terms we get,

$$= P(k_0, \dots, k_s) + P(k_1, \dots, k_s)$$

q.e.d

Next we prove some relationships between the functions a(s), b(s), and c(s) for $0 \leq s \leq t$.

Proposition 1

$$a(s) = k_s b(s-1) + a(s-2), \quad c(s) = k_s c(s-1) + c(s-2), \quad \textit{and} \quad b(s) = k_s a(s-1) + b(s-2)$$

Proof: This first equation can easily be verified for $s = 0, 1$. For $s \geq 2$:

$$a(s) - a(s-2) = \sum_{i=0}^{s} k_i P(k_{i+1}, \dots, k_s) P(k_{i+1}, \dots, k_t)$$

$$- \sum_{i=0}^{s-2} k_i P(k_{i+1}, \dots, k_{s-2}) P(k_{i+1}, \dots, k_t)$$

$$= \sum_{i=0}^{s-1} k_i [k_s P(k_{i+1}, \dots, k_{s-1}) + P(k_{i+1}, \dots, k_{s-2})] P(k_{i+1}, \dots, k_t)$$

$$+ k_s P(k_{s+1}, \dots, k_t) - \sum_{i=0}^{s-2} k_i P(k_{i+1}, \dots, k_{s-2}) P(k_{i+1}, \dots, k_t)$$

$$= \sum_{i=0}^{s-1} k_i k_s P(k_{i+1}, \dots, k_{s-1}) P(k_{i+1}, \dots, k_t) + k_s P(k_{s+1}, \dots, k_t) + k_{s-1} P(k_s, \dots, k_{s-2})$$

$$= k_s[a(s-1) + P(k_{s+1}, \ldots, k_t)]$$

$$= k_s b(s-1).$$

A similar method yields $c(s) = k_s c(s-1) + c(s-2)$ and $b(s) = k_s a(s-1) + b(s-2)$.

q.e.d

So far, we have assumed that $k_0, \ldots, k_t$ are variables. From now on, following the Euclidean algorithm, we consider them as real numbers such that $k_0 \geq 0$ and $k_1, \ldots, k_t > 0$. This means that $a(s) \geq 0$, $b(s) > 0$, and $c(s) \geq 1$ for $-2 \leq s \leq t$. Our next goal is to show that $\frac{a(t)}{c(t)} \geq \frac{a(s)}{c(s)}$ for $s \leq t$.

Proposition 2 $\frac{a(t)}{c(t)}$ *is the maximum of the series,* $\{\frac{a(t)}{c(t)}, \frac{a(t-1)}{c(t-1)}, \frac{a(t-2)}{c(t-2)}, \ldots, \frac{a(-2)}{c(-2)}\}$

In order to do this, we consider the determinant of the matrix:

$$det\begin{pmatrix} a(s-2) & b(s-1) \\ c(s-2) & c(s-1) \end{pmatrix}.$$

Lemma 4

$$det\begin{pmatrix} a(s-2) & b(s-1) \\ c(s-2) & c(s-1) \end{pmatrix} = -b(-1) \quad or \quad -b(-2)$$

for s even or s odd, respectively.

Proof:

$$det\begin{pmatrix} a(s-2) & b(s-1) \\ c(s-2) & c(s-1) \end{pmatrix} = det\begin{pmatrix} a(s-2) & k_s a(s-2) + b(s-3) \\ c(s-2) & k_s c(s-2) + c(s-3) \end{pmatrix}$$

$$= det\begin{pmatrix} a(s-2) & b(s-3) \\ c(s-2) & c(s-3) \end{pmatrix}$$

Continuing this process of substitution reduces the determinant to:

$$det\begin{pmatrix} a(-2) & b(-1) \\ c(-2) & c(-1) \end{pmatrix} = -b(-1)$$

for s even, and

$$det \begin{pmatrix} a(-1) & b(-2) \\ c(-1) & c(-2) \end{pmatrix} = -b(-2)$$

for s odd.

q.e.d

Because $b(-1)$ and $b(-2)$ are always greater than 0, we have shown that

$$a(s-2)c(s-1) - b(s-1)c(s-2) = -b(-1) \quad or \quad -b(-2) < 0.$$

Therefore $a(s-2)c(s-1) < b(s-1)c(s-2)$. Dividing both sides by $c(s-1)c(s-2)$, we get

$$\frac{b(s-1)}{c(s-1)} > \frac{a(s-2)}{c(s-2)}.$$

Recall if $ad - bc < 0$ then $\frac{a+bx}{c+dx}$ is a strictly increasing function of x. Therefore we know that

$$\frac{a(s-2) + jb(s-1)}{c(s-2) + jc(s-1)}$$

is a strictly increasing function for j. Setting $j = k_s$

$$\frac{a(s-2)}{c(s-2)} < \frac{a(s-2) + k_s b(s-1)}{c(s-2) + k_s c(s-1)} = \frac{a(s)}{c(s)}$$

Similarly,

$$\frac{a(s-3)}{c(s-3)} < \frac{a(s-3) + k_s b(s-2)}{c(s-3) + k_s c(s-2)} = \frac{a(s-1)}{c(s-1)}$$

Thus, we obtain

$$\frac{a(s)}{c(s)} > \frac{a(s-2)}{c(s-2)} > \frac{a(s-4)}{c(s-4)} > \dots and \quad \frac{a(s-1)}{c(s-1)} > \frac{a(s-3)}{c(s-3)} > \frac{a(s-5)}{c(s-5)} > \dots$$

Next, consider $det \begin{pmatrix} a(t) & a(t-1) \\ c(t) & c(t-1) \end{pmatrix}$

(Recall that $a(t) = b(t)$). Using the same methods, the determinant reduces to $b(-1)$ or $b(-2)$ for t odd or t even, respectively. Thus, $a(t)c(t-1) > a(t-1)c(t)$. Again, dividing by $c(t)c(t-1)$ yields $\frac{a(t)}{c(t)} > \frac{a(t-1)}{c(t-1)}$. Thus

$$\frac{a(t)}{c(t)} > \frac{a(t-1)}{c(t-1)} > \frac{a(t-3)}{c(t-3)} > \dots and \quad \frac{a(t)}{c(t)} > \frac{a(t-2)}{c(t-2)} > \frac{a(t-4)}{c(t-4)} > \dots$$

Therefore, $\frac{a(t)}{c(t)}$ is the maximum of the series (we will need the fact that $\frac{c(t)}{a(t)}$ is the minimum later in the argument).

q.e.d.

5 Proof of the Theorem

Theorem 2 *For any positive integers p and q with $p \geq q$ and $gcd(p, q) = 1$, the Log Canonical Threshold of the curve $x^p + y^q$ is $\frac{1}{p} + \frac{1}{q}$*

5.1 Blowups

Before examining the case for arbitrary p and q as well as an lengthy example, we introduce two practical rules:

1) When blowing up, we always substitute for the variable in the proper transform of lower degree. Consider $x^a y^b (x^c + y^d)$ with $a, b, c, d \geq 0$ and $c \geq d$. Substituting $x = ty$ gives

$$(ty)^a y^b ((ty)^c + y^d) \Rightarrow y^{a+b+d} t^a (t^c y^{c-d} + 1)$$

As we saw earlier this is not allowed because $E \cap T = \varnothing$. Thus, our only option is to substitute $y = ux$.

Consider $x^{p_0} + y^{p_1}$ with $p_0 \geq p_1$. Following rule (1) $x^{p_0} + y^{p_1}$ is blown up (substituting $y = y_1 x, y_1 = y_2 x$, and so on) until the proper transform is $x^{p_0 - k_0 p_1} + y_{k_0}^{p_1}$. At this point we switch the variable we blowup. Instead of blowing up y we blowup x k_1 times where k_1 is the smallest integer such that $p_1 - k_1 p_2 \leq p_2$. We continue this process until the proper transform and the exceptional divisor have normal crossing (i.e. the proper transform is of the form $1 + x_\alpha$ or $1 + x_\beta$).

the general case with $p = p_0$ and $q = p_1$ follows the Euclidean Algorithm:

$$p_0 = k_0 p_1 + p_2$$

$$p_1 = k_1 p_2 + p_3$$

$$\vdots$$

$$p_{t-1} = k_{t-1} p_t + p_{t+1}$$

$$p_t = k_t$$

The total number of blowups necessary to resolve the singularity is therefore equal to $k_0 + \ldots + k_t$. We keep one variable k_0 times, then switch to keeping the other variable k_1 times, and so forth. Note that $p_i \geq p_{i+1}$ for all $0 \leq i \leq t$ and $p_{t+1} = 1$.

2) At each stage of the blowup process, there is a new exceptional divisor. We set the variable with the highest degree in the term that is factored out of the proper transform to be the exceptional divisor.

Continuing with the general case, we now look at the multiplicity of the exceptional divisor at each stage of the blowup. Let A_i be the multiplicity of the exceptional divisor after the i^{th} blowup:

$$x^{p_0} + y^{p_1}$$

after k_0 blowups (substituting as before for $y = y_1 x$, $y_1 = y_2 x$, and so on) we get,

$$x^{k_0 p_1}(x^{p_0 - k_0 p_1} + (y_{k_0})^{p_1}) = x^{k_0 p_1}(x^{p_2} + (y_{k_0})^{p_1})$$

so $A_{k_0} = k_0 p_1$.
Next we switch (i.e. blowup $x = x_1 y_{k_0}$) and get,

$$(y_{k_0})^{p_0}(x_1)^{k_0 p_1}((x_1)^{p_2} + (y_{k_0})^{p_1 - p_2})$$

After a total of $k_0 + k_1$ blowups we get,

$$(y_{k_0})^{k_1 p_0}(x_{k_1})^{k_0 p_1}((x_{k_1})^{p_2} + (y_{k_0})^{p_1 - k_1 p_2}) = (y_{k_0})^{k_1 p_0}(x_{k_1})^{C_{k_0}}((x_{k_1})^{p_2} + (y_{k_0})^{p_3})$$

so $A_{k_0 + k_1} = k_1 p_0$.

Since we will only be interested in A_s with $s = k_0 + \ldots + k_j$ for some integer $j \leq t$ we streamline the argument and set $A_{k_0 + \ldots + k_s} = A(s)$.

After $k_0 + \ldots + k_{s-1}$ blowups the exceptional divisor and proper transform look like:

$$(x_\alpha)^{A(s-1)}(y_\beta)^{A(s-2)}((x_\alpha)^{p_s} + (y_\beta)^{p_{s+1}})$$

Where α, β index the number of substitutions for x and y respectively. After k_s more blowups,

$$(x_\alpha)^{k_s[A(s-1) + p_{s+1}] + A(s-2)}(y_{\beta + k_s})^{A(s-2)}((x_\alpha)^{p_{s+2}} + (y_{\beta + k_s})^{p_{s+1}})$$

So we get a recursive formula for the multiplicity of the exceptional divisor:

$$A(s) = k_s[A(s-1) + p_{s+1}] + A(s-2)$$

Let us consider the example $x^{30} + y^{13}$. Each new line corresponds to a blowup in the progression:

$$x^{30} + y^{13}$$

$$30 = \underline{2} \cdot 13 + 4 \qquad \begin{cases} x^{13}(x^{17} + (y_1)^{13}), & y = y_1 x \\ x^{26}(x^4 + (y_2)^{13}), & y_1 = y_2 x \end{cases}$$

$$13 = \underline{3} \cdot 4 + 1 \qquad \begin{cases} (x_1)^{26}(y_2)^{30}((x_1)^4 + (y_2)^9), & x = x_1 y_2 \\ (x_2)^{26}(y_2)^{60}((x_2)^4 + (y_2)^5), & x_1 = x_2 y_2 \\ (x_3)^{26}(y_2)^{90}((x_3)^4 + y_2), & x_2 = x_3 a_2 \end{cases}$$

$$4 = \underline{4} \cdot 1 \qquad \begin{cases} (x_3)^{117}(y_3)^{90}((x_3)^3 + y_3), & y_2 = y_3 x_3 \\ (x_3)^{208}(y_4)^{90}((x_3)^2 + y_4), & y_3 = y_4 x_3 \\ (x_3)^{299}(y_5)^{90}(x_3 + y_5), & y_4 = y_5 x_3 \\ (x_3)^{390}(y_6)^{90}(1 + y_6), & y_5 = y_6 x_3 \end{cases}$$

Looking at the Euclidian algorithm expansion for $\frac{30}{13} = 2 + \frac{1}{3+\frac{1}{4}} = \langle 2, 3, 4 \rangle$:

$$30 = 2 \cdot 13 + 4$$

$$13 = 3 \cdot 4 + 1$$

$$4 = 4 \cdot 1$$

The x term stays the exceptional divisor for 2 blowups, the y_2 term stays for 3 blowups, and x_3 for 4 blowups. To check the recursive formula,

$$A(3) = k_s[A(2) + p_4] + A(1) \Rightarrow 390 = (4)[(90) + (1)] + (26)$$

5.2 Differential Forms

Now for the general case $x^{p_0} + y^{p_1}$. Let C_i be the multiplicity of the exceptional divisor of the differential form after the i^{th} blowup. We make the same substitutions we did before:

$$dx \wedge dy$$

after k_0 blowups we get,

$$x^{k_0} dx \wedge d(y_{k_0})$$

so $C_{k_0} = k_0$.
Next we switch and get

$$(y_{k_0})^{k_0+1}(x_1)^{k_0} d(x_1) \wedge d(y_{k_0})$$

After a total of $k_0 + k_1$ blowups we get,

$$(y_{k_0})^{k_1 k_0 + k_1}(x_{k_1})^{k_0} d(x_{k_1}) \wedge d(y_{k_0}) = (y_{k_0})^{k_1 k_0 + k_1}(x_{k_1})^{C_{k_0}} d(x_{k_1}) \wedge d(y_{k_0})$$

so $C_{k_0+k_1} = k_1 k_0 + k_1$.

In a similar fashion to before, set $C_{k_1+\ldots+k_s} = C(s)$. After $k_1 + \ldots + k_{s-1}$ blowups the differential form looks like:

$$(x_\alpha)^{C(s-1)}(y_\beta)^{C(s-2)} d(x_\alpha) \wedge d(y_\beta)$$

With α, β defined as before. After k_s more blowups,

$$(x_\alpha)^{k_s[C(s-1)+1]+C(s-2)}(y_\beta + k_s)^{C(s-2)} d(x_\alpha) \wedge d(y_\beta + k_s)$$

This gives rise to the recursive formula:

$$C(s) = k_s[C(s-1)+1] + C(s-2)$$

Let's consider the example $x^{30} + y^{13}$:

$$dx \wedge dy$$

2 substitutions
$$\begin{cases} x dx \wedge dy_1, & y = y_1 x \\ x^2 dx \wedge dy_2, & y_1 = y_2 x \end{cases}$$

switch

3 substitutions
$$\begin{cases} (x_1)^2(y_2)^3 dx_1 \wedge dy_2, & x = x_1 y_2 \\ (x_2)^6(y_2)^2 dx_2 \wedge dy_2, & x_1 = x_2 y_2 \\ (x_3)^9(y_2)^2 dx_3 \wedge dy_2, & x_2 = x_3 y_2 \end{cases}$$

switch

4 substitutions
$$\begin{cases} (x_3)^{12}(y_3)^9 dx_3 \wedge dy_3, & y_2 = y_3 x_3 \\ (x_3)^{22}(y_4)^9 dx_3 \wedge dy_4, & y_3 = y_4 x_3 \\ (x_3)^{32}(y_5)^9 dx_3 \wedge dy_5, & y_4 = y_5 x_3 \\ (x_3)^{42}(y_6)^9 dx_3 \wedge dy_6, & y_5 = y_6 x_3 \end{cases}$$

To check the recursive formula,

$$C(3) = k_3[C(2) + 1] + C(1) \Rightarrow 42 = (4)[(9) + 1] + (2)$$

5.3 Applying the Algebra

In the following discussion we set $p_0 = P(k_0, \ldots, k_t)$, $p_1 = P(k_0, \ldots, k_{t-1})$, $p_2 = P(k_0, \ldots, k_{t-2}), \ldots, p_{t+1} = P_{-1}$. As stated previously, we introduced $P(k_0, \ldots, k_t)$ to parallel the Euclidean algorithm so this change is natural.

Proposition 3 *$A(s) = a(s)$ for all $s \leq t$*

We have seen that the following recursion exists for our A terms:

$$A(s) = k_s[A(s-1) + P(k_{s+1}, \ldots, k_t)] + A(s-2)$$

Using our relationships between $a(s)$ and $b(s)$ obtained in proposition 1: $a(s) = k_s b(s-1) + a(s-2)$ and $b(s) = a(s) + P(k_{s+2}, \ldots, k_t)$ we can write

$$a(s) = k_s[a(s-1) + P(k_{s+1}, \ldots, k_t)] + a(s-2)$$

Notice that this has the same recursive formula as $A(s)$, thus if we show that the starting term for each function is the same, $a(s)$ and $A(s)$ are equivalent.

for $s = 0$,

$$A(0) = k_0(A(-1) + P(k_1, k_2, \ldots, k_t)) + A(-2)$$

$$= k_0 P(k_1, \ldots, k_t)$$

$$a(0) = k_0(a(-1) + P(k_1, \ldots, k_t)) + a(-2)$$
$$= k_0 P(k_1, \ldots, k_t)$$

Thus these functions are equivalent and must therefore be equal for every $s \leq t$.

q.e.d

Recall that $a(t) = P(k_1, \ldots, k_t)P(k_0, \ldots, k_t) = p_0 p_1$. Therefore, it is necessarily true that $A(t) = p_0 p_1$.

Next, we draw a similar parallel between $C(s)$ and $c(s)$.

Proposition 4 *$C(s) = c(s)$ for all $s \leq t$*

Recall $C(s) = k_s(C(s-1)+1) + C(s-2)$,
and $c(s) = k_s c(s-1) + c(s-2)$.

Instead of $c(s)$, let us look at $c(s) - 1$.

$$c(s) - 1 = k_s c(s-1) + c(s-2) - 1$$

So $[c(s)-1] = k_s([c(s-1)-1]+1) + [c(s-2)-1]$

Again, if we can show that the $s = 0$ terms of each function are equal, then they must be equivalent.

$$c(0) - 1 = k_0 c(-1) + c(-2) - 1$$

$$= k_0 \quad \text{since} \quad c(-1) = c(-2) = 1$$

$$C(0) = k_0(C(-1)+1) + C(-2)$$

$$= k_0 \quad \text{since} \quad C(-1) = C(-2) = 0$$

q.e.d

We know that $c(t) = P(k_0, \ldots, k_t) + P(k_1, \ldots, k_t) = p_0 + p_1$ so $c(t) - 1 = p_0 + p_1 - 1 \Longrightarrow C(t) = p_0 + p_1 - 1$

The value in question is the log canonical threshold $\alpha = min\{\frac{C(i)+1}{A(i)}\}$. We now know that this value can also be written as $\alpha = min\{\frac{(c(i)-1)+1}{a(i)}\}$. We proved previously that s=t is the minimum value for $\frac{c(s)}{a(s)}$ thus, for any singularity of the form $x^{p_0} + y^{p_1}$ has log canonical threshold

$$\alpha = \frac{c(t)}{a(t)} = \frac{p_0 + p_1}{p_0 p_1} = \frac{1}{p_0} + \frac{1}{p_1}$$

corollary 3 *The Theorem holds true for any positive integers p and q with $gcd(p,q)$ not necessarily equal to 1.*

Proof:
coming soon...

6 References:

Brieskorn, E. *Plane Algebraic Curves.*
Last Year's REU
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others to be added...