Math 499/699: VIGRE Senior Seminar

Let $f_1, \ldots, f_{n+1}$ be $n+1$ polynomials in n variables. The resultant is a polynomial in the coefficients of the f_i which vanishes whenever the f_i have a common zero.

The problem of finding explicit formulas for resultants is classical. Sylvester and Bézout developed determinantal formulas, i.e. formulas that express the resultant as the determinant of some matrix, for $n = 1, 2, 3$ when the $n+1$ polynomials all have the same degree.

For example, let $n = 1$ and consider $p(x) = a_0x^2 + a_1x + a_2 = 0$ and $q(x) = b_0x^3 + b_1x^2 + b_2x + b_3 = 0$. In matrix form we have:

$$\begin{pmatrix} 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} x^3 \\ x^2 \\ x \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since square matrices have some very nice properties, we translate this into an equivalent system involving a square matrix. With respect to common zeroes, it will not matter if equations like $x^ip(x) = 0$ and $x^jq(x) = 0$ are added to the system. So, in this particular case, we can add $xp(x) = 0$, $x^2p(x) = 0$ and $xq(x) = 0$ which gives the following matrix equation:

$$\begin{pmatrix} 0 & 0 & a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 & a_2 & 0 \\ a_0 & a_1 & a_2 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 \\ b_0 & b_1 & b_2 & b_3 & 0 \end{pmatrix} \begin{pmatrix} x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Such a square matrix equation has nontrivial solutions only when the determinant of the matrix is zero. This system has a solution if $p(x)$ and $q(x)$ have a common solution. Consequently, the determinant will vanish whenever $p(x)$ and $q(x)$ have a common solution.

When the $n+1$ polynomials do not have the same degree, the situation was less understood classically. Note that the procedure given above for $n = 1$ doesn't care about the degrees of the two polynomials; however, the results of Sylvester and Bézout do not extend to $n = 2, 3$. Just recently, Amit Khetan has given formulas for the $n = 2, 3$ cases.

For $n = 2$ Khetan's theorem is that the resultant of a system $f_1, f_2, f_3 \in \mathbb{C}[x_1, x_2, x_3, x_1^{-1}, x_2^{-1}]$ with common Newton polygon Q is the determinant of the block matrix:

$$\begin{pmatrix} B & L \\ \widetilde{L} & 0 \end{pmatrix}$$

where L and $\widetilde{L}$ are linear terms and the entries of B are cubic forms in the coefficients of the polynomials.

In this seminar, we will develop the mathematics behind Khetan's result. In particular, we will cover spectral sequences, the Koszul complex, the Tate resolution, Beilinson resolution and spectral sequence, etc.