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Let $X = \{(x,y) : y^2 = x^2 + x^3\} \subset \mathbb{A}^2(\mathbb{C})$ be the nodal cubic curve. Then there is a surjective group homomorphism

$$\mathrm{Pic}(X) \to \mathbb{C}^*.$$

We use the parametrization:

$$\begin{array}{rcl} F : \mathbb{A}^1(\mathbb{C}) & \to & X \\ s & \mapsto & (s^2-1, s^3-s) \end{array}$$

Step I: I left out half of what I needed from the Moving Lemma:

Let D be a Cartier divisor on X. Then there is a principal divisor $\mathrm{div}(g)$ such that $E = D - \mathrm{div}(g)$ satisfies

- if (U_α, g_α) is a local representative of E at 0 then g_α is invertible at $s = \pm 1$;
- E has empty support, i.e., $[E] = 0$.

proof: Express $D = \{(U_\alpha, f_\alpha)\}$ for some covering $\{U_\alpha\}$ of X. Suppose U_α contains 0 and express

$$f_\alpha = (s-1)^{e_1}(s+1)^{e_{-1}}h$$

where $h \in \mathbb{C}(s)$ is invertible at $s = \pm 1$. Replacing D by $D' = D - \mathrm{div}((s-1)^{e_1}(s+1)^{e_{-1}})$, we obtain the first condition.

Now consider the associated Weil divisor

$$[D'] = \sum_i e_i z_i, \quad z_i \in X \setminus \{0\} = \mathbb{A}^1(\mathbb{C}) \setminus \{s = \pm 1\}.$$

Writing $\tilde{h} = \prod_i (s - z_i)^{e_i}$, our desired divisor is

$$E = D' - \mathrm{div}(\tilde{h}). \ \square$$

Step II: There exists a well-defined surjective homomorphism

$$\begin{array}{rcl} \phi : \mathrm{Pic}(X) & \to & \mathbb{C}^* \\ D & \mapsto & g_\alpha(1)/g_\alpha(-1), \end{array}$$

where $E = \{(U_\alpha, g_\alpha)\}$ is a representative of D in $\mathrm{Pic}(X)$ as described in Step I and $U_\alpha \ni 0$.

To prove this is well-defined, suppose we have a second representative $E' = \{(U_\alpha, g'_\alpha)\}$ of the class of D. Then there exists $h \in \mathbb{C}(s)$ such that $g'_\alpha = hg_\alpha$, modulo units on U_α. The function h has no zeros or poles on $\mathbb{A}^1(\mathbb{C})$, hence is constant. Thus $g_\alpha(1)/g_\alpha(-1) = g'_\alpha(1)/g'_\alpha(-1)$ and ϕ is independent of the representation.