February 13, 2008

1)Let X be a variety and $E \to X$ a vector bundle of rank r. Suppose that $s \in \Gamma(X, E)$ is a global section of E such that

$$s^{-1}(0) = \{x \in X : s(x) = 0\}$$

has pure codimension r. Prove that

$$[s^{-1}(0)] = c_r(E) \cap [X],$$

i.e., the top Chern class of E gives the class where a section vanishes. *Hint:* Use example 3.3.2 from Fulton!

2)Let Q be the universal quotient bundle on the Grassmannian $\mathrm{Gr}(2, V)$ parametrizing two-dimensional subspaces W of a four-dimensional vector space V. Show that $c_1(Q) \cap \mathrm{Gr}(2, V)$ represents the divisor class

$$\{W \in \mathrm{Gr}(2, V) : W \cap L_2 \neq 0\}$$

where $L_2 \in \mathrm{Gr}(2, V)$. Verify this is linearly equivalent to the hyperplane class of $\mathrm{Gr}(2, V)$ with respect to the Plücker embedding

$$\mathrm{Gr}(2, V) \hookrightarrow \mathbb{P}(\bigwedge^2 V) \simeq \mathbb{P}^5.$$

Show that $c_2(Q) \cap \mathrm{Gr}(2, V)$ represents the locus

$$\{W \in \mathrm{Gr}(2, V) : W \supset L_1\}$$

where $L_1 \subset V$ is a subspace of dimension one. Conclude that

$$\deg(c_1(Q)^4 \cap \mathrm{Gr}(2, V)) = 2, \quad \deg(c_1(Q)^2 c_2(Q) \cap \mathrm{Gr}(2, V)) = 1,$$
$$\deg(c_2(Q)^2 \cap \mathrm{Gr}(2, V)) = 1.$$

3)Let S be the universal subbundle on the $\mathrm{Gr}(2, 4)$. Show that the degree of $c_4(\mathrm{Sym}^3 S^*)$ is equal to 27. Explain how this is related to the problem of enumerating the lines on a cubic surface.

4)Let V be a finite-dimensional vector space over a field of characteristic zero. Let $\mathbb{P}(\mathrm{Sym}^d V^*)$ denote the projective space parametrizing degree d hypersurfaces in $\mathbb{P}(V)$; for each $F \neq 0 \in \mathrm{Sym}^d V^*$, let $X(F) \subset \mathbb{P}(V)$ denote the associated hypersurface. Consider the incidence variety

$$Z = \{(v, F) : v \text{ is a singular point of } X(F)\} \subset \mathbb{P}(V) \times \mathbb{P}(\mathrm{Sym}^d V^*).$$

Realize $Z = \mathbb{P}(E)$ for some vector bundle E on $\mathbb{P}(V)$. Prove the image of the projection $\pi_2 : \mathbb{P}(E) \to \mathbb{P}(\mathrm{Sym}^d V^*)$ is a hypersurface; this is called the *discriminant hypersurface*.

5)Retain the notion of the last exerise. When $d = 2$ compute the degree of discriminant hypersurface. Do the same when d is arbitrary but $\dim(V) = 2$. *Challenge:* Try the case $d = 3$ and $\dim(V) = 3$ or 4.

6)Let V be a finite-dimensional vector space. Consider the incidence variety

$$Z = \{(W, F) : \mathbb{P}(W) \subset X(F)\} \subset \mathrm{Gr}(2, V) \times \mathbb{P}(\mathrm{Sym}^d V^*).$$

Realize $Z = \mathbb{P}(E)$ for some vector bundle E on the Grassmannian.

7)Retain the notation of the last problem and assume $\dim(V) = 4$ and $d = 4$. Prove the image of the projection $\pi_2 : Z \to \mathbb{P}(\mathrm{Sym}^d V^*)$ is a hypersurface and compute its degree.