Math 567 Lecture Notes

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4 Morphisms from rational curves to projective varieties

4.1 Morphisms to projective space

We recall the following basic facts:

1. Let $\phi_0, \ldots, \phi_n \in k[x_0, x_1]_d$ be homogeneous of degree d, with at least one $\phi_i \neq 0$. Then the rule
$$[x_0, x_1] \mapsto [\phi_0, \ldots, \phi_n] \tag{1}$$
defines a rational map
$$\phi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n.$$
2. If the ideal
$$\langle \phi_0, \ldots, \phi_n \rangle \subset k[x_0, x_1]$$
is irrelevant, i.e., the ϕ_i have no common factors, then ϕ is a morphism.
3. Each morphism $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^n$ can be expressed in the form (1). We define its degree by
$$\deg(\phi) = \deg(\phi_i) = d.$$
4. Two morphisms
$$\phi, \psi : \mathbb{P}^1 \rightarrow \mathbb{P}^n$$
are equal if and only if there exists $\lambda \in k^*$ with $\phi_i = \lambda \psi_i$ for each i.

We selectively sketch the proofs of these. For Fact 2, consider the open subsets

$$U_i = \{z_i \neq 0\} \subset \mathbb{P}^n \quad V_i = \{\phi_i \neq 0\} \subset \mathbb{P}^1,\ i = 0, \dots, n.$$

The irrelevance assumption means that $\{V_i\}_{i=0}^n$ is an affine covering of $\mathbb{P}^1$. We compute coordinate rings

$$k[U_i] = k[z_0/z_i, z_1/z_i, \dots, z_{i-1}/z_i, z_{i+1}/z_i, \dots, z_n/z_i]$$

and

$$k[V_i] = (k[x_0, x_1][\phi_i^{-1}(x_0, x_1)])_{\deg 0} = \{P/\phi_i^N : P \in k[x_0, x_1]_{dN}\}.$$

We have local morphisms $\phi : U_i \to V_i$ given by the homomorphism

$$\begin{array}{rcl} \phi^* : k[U_i] & \to & k[V_i] \\ z_j/z_i & \mapsto & \phi_j/\phi_i. \end{array}$$

These glue together to give a morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^n$.

For Fact 3, assume for notational simplicity that z_0 is not identically zero on the image of ϕ. Then the quotients

$$z_1/z_0, \dots, z_n/z_o \in k(\mathbb{P}^n)$$

pull back to elements

$$\phi^*(z_1/z_0), \dots, \phi^*(z_n/z_0) \in k(\mathbb{P}^1) = k(x_1/x_0).$$

Express

$$\phi^*(z_i/z_0) = p_i(x_1/x_0)/q_i(x_1/x_0),$$

where p_i, q_i are polynomials without common factors. Homogenizing, we obtain homogeneous $A_i, B_i \in k[x_0, x_1]$ with $\deg A_i = \deg B_i$ and no common factors such that

$$\phi^*(z_i/z_0) = A_i/B_i.$$

Claim: There exist $\phi_0, \dots, \phi_n \in k[x_0, x_1]$, homgeneous of the same degree without common factors, such that

$$A_i/B_i = \phi_i/\phi_0,\ i = 1, \dots, n.$$

Choose $\phi_0 = \gcd(B_1, \ldots, B_n)$ and set $\phi_i = \phi_0 A_i/B_i$; these are all homogeneous polynomials of the same degree. It remains to show that have no common factors. Suppose $R|\phi_i$ for each i and R^e is the largest power dividing ϕ_0. It follows that $R^e|B_j$ for some j and thus R does not divide A_j. Using the equality

$$\phi_j B_j = \phi_0 A_j$$

and the fact that $k[x_0, x_1]$ is a unique factorization domain, we conclude that R does not divide ϕ_j, a contradiction.

Proposition 1 *Let* $\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n, d)$ *denote the set of all degree* d *morphisms from* $\mathbb{P}^1$ *to* $\mathbb{P}^n$*, regarded as a subset*

$$\begin{array}{rcl} \mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n, d) & \subset & \mathbb{P}(k[x_0, x_1]^{\oplus n+1}) \simeq \mathbb{P}^{nd+n+d} \\ \phi & \mapsto & [(\phi_0, \ldots, \phi_n)]. \end{array}$$

Then $\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n, d)$ *is Zariski open in* $\mathbb{P}^{nd+n+d}$*, and thus is a smooth rational variety of dimension* $nd + n + d$*.*

proof: For simplicity, we assume that k is algebraically closed. Consider the locus

$$\mathcal{W} = \{[(\phi_0, \ldots, \phi_n)] : \phi_i \text{ have a common factor } \} \subset \mathbb{P}(k[x_0, x_1]_d^{\oplus n+1}),$$

which is the complement to the set of morphisms. Since our field is algebraically closed, we have

$$\mathcal{W} = \{[(\phi_0, \ldots, \phi_n)] : \text{ there exists } L \in k[x_0, x_1]_1 \text{ with } L|\phi_i, i = 0, \ldots, n\}.$$

We claim this is closed.

Again, we have an incidence variety

$$\mathcal{Z} = \{(L, [(\phi_0, \ldots, \phi_n)]) : L|\phi_0, \ldots, \phi_n\} \subset \mathbb{P}(k[x_0, x_1]_1) \times \mathbb{P}(k[x_0, x_1]_d^{\oplus n+1})$$

and we analyze the first projection

$$\pi_1 : \mathcal{Z} \to \mathbb{P}(k[x_0, x_1]) \simeq \mathbb{P}^1.$$

For each nonzero $L \in k[x_0, x_1]_1$ we have the multiplication map

$$\begin{array}{rcl} \mu_L : k[x_0, x_1]_{d-1}^{\oplus n+1} & \to & k[x_0, x_1]_d^{\oplus n+1} \\ (\psi_0, \ldots, \psi_n) & \mapsto & (L\psi_0, \ldots, L\psi_n), \end{array}$$

whose image is a linear subspace of codimension $n+1$. Thus π_1 is a $\mathbb{P}^{dn+d-1}$-bundle over $\mathbb{P}^1$ and $\mathcal{Z}$ is a projective variety of dimension $dn + d$. The image $\pi_2(\mathcal{Z}) = \mathcal{W}$ is therefore a closed proper subset of $\mathbb{P}^{nd+n+d}$. $\square$

Example 2 Consider $\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^1, 1)$, which is an open subset of $\mathbb{P}^3$. Each morphism can be expressed

$$\phi = [\phi_0, \phi_1] = [a_{00}x_0 + a_{01}x_1, a_{10}x_0 + a_{11}x_1]$$

where $a_{00}a_{11} - a_{01}a_{10} \neq 0$. Then we have

$$\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^1, 1) = \mathbb{P}^3 - \{a_{00}a_{11} - a_{10}a_{01} = 0\}$$

the complement of a smooth quadric hypersurface in $\mathbb{P}^3$. This can be identified with the projective linear group PGL_2.

Example 3 Consider $\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^1, 2) \subset \mathbb{P}^5$, with elements of the form

$$\phi = [A_{00}x_0^2 + A_{01}x_0x_1 + A_{11}x_1^2, B_{00}x_0^2 + B_{01}x_0x_1 + B_{11}x_1^2]$$

such that the resultant

$$\det \begin{pmatrix} A_{00} & A_{01} & A_{11} & 0 \\ 0 & A_{00} & A_{01} & A_{11} \\ B_{00} & B_{01} & B_{11} & 0 \\ 0 & B_{00} & B_{01} & B_{11} \end{pmatrix} \neq 0.$$

Exercise 4 Show that $\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^1, d)$ is an affine variety for each d. *Hint:* Use resultants!

4.2 Morphisms to arbitrary projective varieties

Proposition 1 expresses the set of all morphisms

$$\phi : \mathbb{P}^1_{x_0,x_1} \to \mathbb{P}^n_{z_0,\ldots,z_n}$$

can be expressed as an open subset

$$\begin{array}{rcl} \mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n, d) & \subset & \mathbb{P}(k[x_0, x_1]^{\oplus n+1}) \simeq \mathbb{P}^{nd+n+d} \\ \phi & \mapsto & [(\phi_0, \ldots, \phi_n)]. \end{array}$$

The coordinate functions ϕ_i together define a linear transformation

$$\phi^{*,1} : k[z_0, \ldots, z_n]_1 \to k[x_0, x_1]_d$$

and a homomorphism of graded k-algebras

$$\phi^* : k[z_0, \ldots, z_n] \to k[x_0, x_1].$$

From this point of view, it is more natural to interpret the ambient $\mathbb{P}^{nd+n+d}$ as

$$\mathbb{P}(\mathrm{Hom}(k[z_0, \ldots, z_n]_1, k[x_0, x_1]_d)).$$

Suppose now that $X \subset \mathbb{P}^n$ is a projective variety. How do we define the morphisms

$$\phi : \mathbb{P}^1 \to X \subset \mathbb{P}^n?$$

Special case of hypersurfaces: Suppose that $X = \{F = 0\}$ with $F \in k[z_0, \ldots, z_n]_e$. Consider the induced linear transformations:

$$\begin{tikzcd}
\mathrm{Sym}^e\phi^{*,1} : k[z_0, \ldots, z_n]_e = \mathrm{Sym}^e k[z_0, \ldots, z_n]_1 \arrow[r] \arrow[dr, "\phi^{*,e}"'] & \mathrm{Sym}^e k[x_0, x_1]_d \arrow[d, "\text{multiply}"] \\
 & k[x_0, x_1]_{de}
\end{tikzcd}$$

Clearly $\phi(\mathbb{P}^1) \subset X(F)$ if and only if

$$\phi^{*,e} F = F(\phi_0(x_0, x_1), \ldots, \phi_n(x_0, x_1)) = 0.$$

We therefore define

$$\begin{aligned} \mathrm{Mor}(\mathbb{P}^1, X(F), d) &= \{\phi \in \mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n, d) : F(\phi_0(x_0, x_1), \ldots, \phi_n(x_0, x_1)) = 0\} \\ &\subset \mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n, d). \end{aligned}$$

We verify this is closed by analyzing its defining equations: Express each

$$\phi_i = A_{i0} x_0^d + \ldots + A_{id} x_1^d$$

and expand

$$F(\phi_0, \ldots, \phi_n) = \sum_{i=0}^{de} c_i(A_{00}, \ldots, A_{nd}) x_0^i x_1^{de-i}$$

where each $c_i \in k[A_{00}, \ldots, A_{nd}]_e$. This is identically zero (as a function in x_0 and x_1) if and only if

$$c_0 = c_1 = \ldots = c_{de} = 0,$$

a system of $de + 1$ homogeneous equations in $dn + d + n + 1$ unknowns.

Definition 5 Let $X(F) \subset \mathbb{P}^n$ be a hypersurface of degree e. The *expected dimension* of $\mathrm{Mor}(\mathbb{P}^1, X(F), d)$ is defined

$$(nd + n + d) - (de + 1) = nd - ed + d + n - 1.$$

For example, when $d = 1$ the expected dimension is

$$2n - e = \mathrm{expdim} F_1(X(F)) + 3.$$

The three extra parameters express the fact that $\mathrm{Mor}(\mathbb{P}^1, X(F), 1)$ corresponds to lines $\ell \subset X(F)$ along with the choice of an isomorphism $\mathbb{P}^1 \xrightarrow{\sim} \ell$. As we have seen, these isomorphisms are indexed by PGL_2.

General case: Given an arbitrary projective $X \subset \mathbb{P}^n$, we may express X as an intersection of hypersurfaces

$$X = X(F_1) \cap \ldots \cap X(F_r).$$

We therefore define

$$\mathrm{Mor}(\mathbb{P}^1, X, d) = \cap_{i=1}^{r} \mathrm{Mor}(\mathbb{P}^1, X(F_i), d).$$

Exercise 6 Verify this is independent of the choice of expression of X as an intersection of hypersurfaces.

4.3 Group actions on the space of morphisms

Consider composition of morphisms

$$\begin{array}{rcl} \mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^1, d) \times \mathrm{Mor}(\mathbb{P}^1, X, c) & \rightarrow & \mathrm{Mor}(\mathbb{P}, X, dc) \\ (\phi, A) & \mapsto & \phi \circ A. \end{array}$$

Exercise 7 Verify this is a morphism of algebraic varieties; it suffices to do the case $X = \mathbb{P}^n$.

In the special case $c = 1$ we can identify

$$\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^1, 1) = \mathrm{PGL}_2$$

and thus get a (right) group action

$$\begin{array}{rcl} \mathrm{Mor}(\mathbb{P}^1, X, d) \times \mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^1, 1) & \rightarrow & \mathrm{Mor}(\mathbb{P}^1, X, d) \\ \phi \cdot A & = & \phi \circ A. \end{array}$$

Two morphisms $\phi, \psi : \mathbb{P}^1 \rightarrow X$ are *equivalent* if they lie in the same PGL_2-orbit.

Definition 8 The *moduli space of maps* is defined as the set of equivalence classes of morphisms, i.e.,

$$M_{00}(X,d) = \mathrm{Mor}(\mathbb{P}^1, X, d)/\mathrm{PGL}_2.$$

We have *not* proven that $M_{00}(X,d)$ admits a natural structure of an algebraic variety. However this can be proven in certain special cases:

Exercise 9 Show that we can identify

$$\begin{array}{rcl} f : M_{00}(\mathbb{P}^n, 1) & \overset{\sim}{\to} & \mathbb{G}(1,n) \\ \phi & \mapsto & \mathrm{Image}(\phi), \end{array}$$

i.e., each map of degree one is determined completely by its image. Exhibit a morphism of algebraic varieties

$$q : \mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n, 1) \to \mathbb{G}(1,n)$$

inducing f i.e., for each $\phi \in \mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^n, 1)$ we have

$$q(\phi) = f(\text{equivalence class of } \phi).$$

Exercise 10 Suppose $k = \mathbb{C}$ or k is algebraically closed with $\mathrm{char}(k) \neq 2$.

a. Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism of degree two. Show that ϕ has two distinct branch points $b_1, b_2 \in \mathbb{P}^1$.

b. For any distinct $b_1, b_2 \in \mathbb{P}^1$ there exists a unique degree-two morphism $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ with these branch points.

c. For each pair of distinct points $b_1, b_2 \in \mathbb{P}^1$ there is a form

$$F = Az_0^2 + Bz_0z_1 + Cz_1^2 \in k[z_0, z_1]_2$$

such that $J(\{b_1, b_2\}) = \langle F \rangle$; this form is unique up to a scalar multiple.

d. Conclude the existence of a bijection

$$\begin{array}{rcl} f : M_{00}(\mathbb{P}^1, 2) & \overset{\sim}{\to} & \mathbb{P}(k[z_0, z_1]_2) \setminus \{B^2 - 4AC = 0\} \\ \phi & \mapsto & [F]. \end{array}$$

e. Write down an explicit morphism of varieties

$$q : \mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^1, 2) \to \mathbb{P}(k[z_0, z_1]_2) \setminus \{B^2 - 4AC = 0\}$$

inducing f.

4.4 Homology classes of morphisms

Here we work over $\mathbb{C}$. Let $\phi : \mathbb{P}^1 \to X$ be a morphism of degree d. The complex projective line is isomorphic to the Riemann sphere

$$\mathbb{P}^1 \simeq S^2$$

which has homology group

$$H_2(\mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}[\mathbb{P}^1],$$

where the generator is the fundamental class. (The *fundamental class* of an oriented compact manifold is Poincaré dual to the class of a point.) Its image

$$\beta := \phi_*[\mathbb{P}^1] \in H_2(X, \mathbb{Z})$$

has degree d, i.e., the interesection with the hyperplane class $\beta \cap H = d$.

The class β is constant for continuous deformations of ϕ. If we define

$$\mathrm{Mor}(\mathbb{P}^1, X, \beta) = \{\phi \in \mathrm{Mor}(\mathbb{P}^1, X, d) : \phi_*[\mathbb{P}^1] = \beta\}$$

then the morphism space is a disjoint union of closed subsets

$$\mathrm{Mor}(\mathbb{P}^1, X, d) = \coprod_{\deg(\beta)=d} \mathrm{Mor}(\mathbb{P}^1, X, \beta).$$

Exercise 11 Let $X \subset \mathbb{P}^3$ be a smooth cubic surface. Enumerate the irreducible components of $\mathrm{Mor}(\mathbb{P}^1, X, 1)$ and compute their dimensions.

4.5 Moduli of stable maps

Assume that the base field is algebraically closed.

Definition 12 Let C be a reduced curve. A point $p \in C$ is a *node* if the tangent cone to C at p is isomorphic to $xy = 0$, i.e., C has two smooth branches at p meeting transversely.

Definition 13 Let C be a nodal curve with irreducible components $C_1, \ldots, C_r$. The *dual graph* is the graph with vertices $v_i, i = 1, \ldots, r$ indexed by the components C_i, and edges $e_{ij}, 1 \leq i < j \leq r$ indexed by the points in the intersection $C_i \cap C_j$. (There may be multiple edges joining two vertices.)

The following combinatorial result is left as an exercise:

Proposition 14 (Exercise) *Let C be a nodal connected projective curve of arithmetic genus zero, i.e., $H^1(C, \mathcal{O}_C) = 0$. Then C is a tree of $\mathbb{P}^1$'s, i.e., each irreducible component of C is isomorphic to $\mathbb{P}^1$ and the dual graph is a tree.*

Definition 15 Let X be a variety. Consider a morphism $\phi : C \to X$ from a nodal connected curve of arithmetic genus zero. It is *stable* if, for each irreducible component $C_i \simeq \mathbb{P}^1 \subset C$ with $\phi|C_i$ constant, three exist (at least) three distinct irreducible components $C_{j_1}, C_{j_2}, C_{j_3} \subset C$ meeting C_i in a node.

Two stable maps $\phi, \psi : C \to X$ are *equivalent* if there exists an automorphism $A : C \to C$ such that $\psi = \phi \circ A$.

Definition 16 Let $X \subset \mathbb{P}^n$ be a projective variety and $d \geq 0$. The *moduli space of stable maps* $\overline{M}_{0,0}(X, d)$ consists of all equivalence classes of stable maps $\phi : C \to X$ with

$$\deg(\phi_*[C]) = \sum_{\text{components } C_i \subset C} \phi_*[C_i] = d.$$

Theorem 17 *[1] $\overline{M}_{0,0}(X, d)$ admits a natural structure as a projective scheme; with respect to this structure, $M_{0,0}(X, d)$ is an open subscheme and the natural assignment*

$$\begin{array}{rcl} q : \mathrm{Mor}(\mathbb{P}^1, X, d) & \to & M_{0,0}(X, d) \subset \overline{M}_{0,0}(X, d) \\ \phi & \mapsto & \textit{equivalence class of } \phi \end{array}$$

is a morphism of schemes.

4.6 Examples of moduli of stable maps

We work over an algebraically-closed field.

Exercise 18 Let $\phi : C \to \mathbb{P}^n$ be a stable map of degree one. Show that $C \simeq \mathbb{P}^1$ and

$$\overline{M}_{0,0}(\mathbb{P}^n, 1) = M_{0,0}(\mathbb{P}^n, 1) = \mathbb{G}(1, n).$$

Exercise 19 Assume that the characteristic is different from two and $\phi : C \to \mathbb{P}^1$ is a stable map of degree two.

a. Show that either $C = \mathbb{P}^1$ and $C = \mathbb{P}^1 \cup_n \mathbb{P}^1$, i.e., two copies of $\mathbb{P}^1$ glued together at a point. In the second case, show that the restrction of ϕ to each copy of $\mathbb{P}^1$ is an isomorphism.

The first case was covered in Exercise 10, where we showed that:

$$\begin{array}{rcl} f : M_{00}(\mathbb{P}^1, 2) & \xrightarrow{\sim} & \mathbb{P}(k[z_0, z_1]_2) \setminus \{B^2 - 4AC = 0\} \\ \phi & \mapsto & \text{equation of branch locus of } \phi \end{array}$$

As for the second case:

b. For each stable map of degree two

$$\phi : \mathbb{P}^1 \cup_n \mathbb{P}^1 \to \mathbb{P}^1, \quad b = \phi(n) \tag{2}$$

show that ϕ is determined uniquely by the branch point b.

c. Show that there is a natural extension

$$\begin{array}{rcl} f : \overline{M}_{00}(\mathbb{P}^1, 2) & \rightarrow & \mathbb{P}(k[z_0, z_1]_2) \\ \phi & \mapsto & \text{equation of branch locus of } \phi. \end{array}$$

If you don't know the formal definition of the branch locus in case (2), you may assume that it has multiplicity two at b.

d. Show there is a well-defined map

$$\begin{array}{rcl} g : \mathbb{P}(k[z_0, z_1]_2) & \rightarrow & \overline{M}_{00}(\mathbb{P}^1, 2) \\ F(z_0, z_1) & \mapsto & (C, \phi) \end{array}$$

where $C = \{y^2 = F(z_0, z_1)\} \subset \mathbb{P}^2$ and

$$\begin{array}{rcl} \phi : C & \rightarrow & \mathbb{P}^1 \\ [y, z_0, z_1] & \mapsto & [z_0, z_1]. \end{array}$$

Verify this is the inverse of f.

We conclude that $\overline{M}_{0,0}(\mathbb{P}^1, 2) \simeq \mathbb{P}^2$.

References

[1] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997.