Math 567 Lecture Notes

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January 17-22, 2007

3 Linear subspaces on a variety

Definition 1 Let $X \subset \mathbb{P}^n$ be a projective variety. We define

$$F_1(X) = \{\ell \in \mathbb{G}(m,n) : \ell \subset X\}.$$

This is sometimes called the 'Fano variety' of X. However, a variety V with ample anticanonical class $-K_V$ is also called a 'Fano variety'. Unfortunately, these usages are sometimes contradictory: There are examples when the 'Fano variety' of linear subspaces has ample *canonical* class.

3.1 Variety structure defined

For background results on varieties of linear subspaces see [1]. The key fact we will need is

Theorem 2 *Let $X \subset \mathbb{P}^n$ be a projective variety (or scheme). Then $F_m(X) \subset \mathbb{G}(m,n)$ is natural a closed subvariety (subscheme).*

proof: **Step 1:** We first consider the special case of a hypersurface of degree d

$$X = \{F = 0\} \subset \mathbb{P}^n, \quad F \in \mathbb{C}[x_0, \dots, x_n]_d.$$

We realize $F_m(X)$ as the locus where a section of a vector bundle on $\mathbb{G}(m,n)$ vanishes.

Recall the exact sequence of vector bundles over $\mathbb{G}(m,n)$

$$0 \to S \to E \to Q \to 0,$$

where E denotes the trivial bundle and S (resp. Q) the universal sub-bundle (resp. quotient-bundle). We dualize the first half

$$E^* \xrightarrow{q} S^* \longrightarrow 0$$

and apply symmetric powers

$$\mathrm{Sym}^d E^* \overset{\mathrm{Sym}^d q}{\longrightarrow} \mathrm{Sym}^d S^* \longrightarrow 0.$$

We may identify

$$\mathrm{Sym}^d E^* = \mathbb{C}[x_0, \dots, x_n]_d \times \mathbb{G}(m,n)$$

and

$$(\mathrm{Sym}^d S^*)_\Lambda = \text{ degree-}d \text{ homogeneous forms on } \Lambda.$$

Exercise 3 Show that

$$\Gamma(\mathbb{G}(m,n), \mathrm{Sym}^d E^*) = \mathbb{C}[x_0, \dots, x_n]_d.$$

Hint: You will need the fact that the Grassmannian is projective, or at least proper.

In particular, we obtain a linear transformation

$$q' : \mathbb{C}[x_0, \dots, x_n]_d = \Gamma(\mathbb{G}(m,n), \mathrm{Sym}^d E^*) \to \Gamma(\mathbb{G}(m,n), \mathrm{Sym}^d S^*).$$

Moreover, $s := q'(F)$ is a section of $\mathrm{Sym}^d S^*$ vanishing at linear subspaces along which F vanishes, i.e.,

$$F_m(X(F)) = \{\ell : s(\ell) = 0\}.$$

It is a general fact that the locus where a section s of a vector bundle $V \to X$ vanishes carries a natural scheme structure. Here is the idea: It suffices to describe this structure locally on X. Suppose V admits a trivialization over U_i

$$\phi_i : V \overset{\sim}{\longrightarrow} k^r \times U_i$$

and $s(1), \dots, s(r)$ are the coordinate components of $\phi_i(s)$. (Here r is the rank of V.) Each $s(j)$ is a regular function on U_i and we see

$$\{s = 0\} \cap U_i = \{s(1) = \dots = s(r) = 0\}.$$

Step 2: We pass to the general case. It suffices to establish the following:

Exercise 4 Let $X \subset \mathbb{P}^n$ be a projective variety (or scheme). Choose generators

$$J(X) = \langle G_1, \ldots, G_r \rangle$$

where each $G_j \in \mathbb{C}[x_0, \ldots, x_n]_{d_j}$ is homogeneous. Show that the intersection

$$\cup_{j=1}^r F_m(X(G_j)) \subset \mathbb{G}(m, n)$$

is independent of the choice of generators.

Thus we may define

$$F_m(X) = \cup_{j=1}^r F_m(X(G_j)).$$

This completes the proof of Theorem 2. $\square$

3.2 General results and examples

Definition 5 Let $F \in \mathbb{C}[x_0, \ldots, x_n]_d$ and $X = X(F)$ a hypersurface of degree d. The *expected dimension* of $F_m(X)$ is

$$\dim \mathbb{G}(m, n) - \mathrm{rank}\mathrm{Sym}^d S^* = (m+1)(n-m) - \binom{d+m}{m}.$$

Proposition 6 *Suppose X is a hypersurface and $F_m(X)$ is of the expected dimension. Then it is a local complete intersection and thus is Cohen-Macaulay and has no embedded points.*

proof: Our interpretation of $F_m(X)$ as the locus where a section of $\mathrm{Sym}^d S^*$ vanishes guarantees that it is locally cut out by $\binom{d+m}{m}$ equations. Our assumption means it has codimension $\binom{d+m}{m}$, so it is a local complete intersection. It is a general fact that any subscheme of affine space of codimension r and defined by r equations

$$Y = \{s(1) = \ldots = s(r) = 0\} \subset \mathbb{A}^N$$

is Cohen-Macaulay [4, pp. 459]. $\square$

Definition 7 Let X be a projective variety. $F_m(X)$ is *smooth* if it is smooth as a scheme.

Example 8 Consider the smooth quadric hypersurface

$$X = \{wx - yz = 0\} \subset \mathbb{P}^3.$$

Set-theoretically, $F_1(X)$ consists of two copies of $\mathbb{P}^1$

$$\{sw - ty = tx - sz = 0 : [s,t] \in \mathbb{P}^1\} \cup \{aw - bz = bx - ay = 0 : [a,b] \in \mathbb{P}^1\}.$$

Recall from elementary geometry that a 'hyperboloid of one sheet' has two transverse rulings.

We analyze the scheme structure over the affine open subset

$$U_{01} \simeq \mathbb{A}^4 \subset \mathbb{G}(1,3).$$

This corresponds to subspaces

$$\mathrm{span}(e_0 + b_{02}e_2 + b_{03}e_3, e_1 + b_{12}e_2 + b_{13}e_3)$$

consisting of vectors of the form

$$c_0e_0 + c_1e_1 + (c_0b_{02} + c_1b_{12})e_2 + (c_0b_{03} + c_1b_{13})e_3.$$

Plugging in

$$w = c_0, x = c_1, y = c_0b_{02} + c_1b_{12}, z = c_0b_{03} + c_1b_{13}$$

into our defining equation, we get

$$b_{02}b_{03} = b_{12}b_{13} = b_{02}b_{13} + b_{12}b_{03} - 1 = 0.$$

The resulting ideal is

$$\langle b_{02}, b_{13}, b_{12}b_{03} - 1\rangle \cap \langle b_{03}, b_{12}, b_{02}b_{13} - 1\rangle$$

which is the union of two rational curves (with multiplicity one).

The following example shows that while the lines on a hypersurface have no embedded points, they need not be reduced:

Example 9 We repeat the analysis of Example 8 with

$$X = \{x^2 - yz = 0\} \subset \mathbb{P}^3.$$

The lines are of the form

$$\{tx - sz = sx - ty = 0\}$$

which is parametrized by $\mathbb{P}^1$. In this case, the defining equations over U_{01} are

$$b_{02}b_{03} = b_{12}b_{13} - 1 = b_{02}b_{13} + b_{03}b_{12} = 0.$$

These imply

$$b_{03} + b_{02}b_{13}^2 = 0$$

and thus

$$b_{02}^2 b_{13}^2 = 0.$$

Since $b_{13} \neq 0$ we obtain

$$b_{02}^2 = 0, \quad b_{12}b_{13} = 1.$$

This is a rational curve with multiplicity two.

See [2] [3] for recent results on the dimension of the space of linear subspaces on a hypersurface.

3.3 Tangent spaces to $F_m(X)$

Theorem 10 *Let $X \subset \mathbb{P}^n$ be projective and $\mathbb{P}(\Lambda) = \ell \subset X$ a linear subspace corresponding to $[\ell] \in F_m(X)$. For each $p \neq 0 \in \mathbb{C}^{n+1}$, write*

$$\mathbb{T}_p X = \{[a_0, \ldots, a_n] : \sum_{i=0}^{n} \partial F/\partial x_i|p \cdot a_i = 0 \text{ for each } F \in J(X)\}.$$

Then the subspace

$$T_{[\ell]}F_m(X) \subset T_{[\Lambda]}\mathrm{Gr}(m+1, n+1)$$

is given by

$$\{\phi \in \mathrm{Hom}(\Lambda, \mathbb{C}^{n+1}/\Lambda) : \phi(p) \in \mathbb{T}_p X \text{ for each } p \in \Lambda\}.$$

proof: [5, pp. 209]

Exercise 11 Let $X \subset \mathbb{P}^n$ denote a smooth quadric hypersurface. Show that $F_1(X)$ is smooth. *Challenge:* How about $F_m(X)$ for $m > 1$?

Exercise 12 Let $X \subset \mathbb{P}^3$ be the quadric cone

$$X = \{[w, x, y, z] : x^2 - yz\}$$

and $\ell = \{x = y = 0\} \subset X$. Compute $T_\ell F_1(X)$ using the characterization given above.

For cubic hypersurface, there are also some very strong results:

Theorem 13 *[1] Let $X \subset \mathbb{P}^n$ be a smooth cubic hypersurface. Then $F_1(X)$ is smooth of the expected dimension.*

proof: Let $\ell \subset X$ be a line on X. Choose coordinates such that $\ell = \{x_2 = \ldots = x_n = 0\}$ and x_0, x_1 are coordinates on ℓ. If $X = \{F = 0\}$ then we can expand

$$F = G + x_0 Q_0 + x_1 Q_1 + x_0^2 L_{00} + x_0 x_1 L_{01} + x_1^2 L_{11}$$

with

$$G, Q_0, Q_1, L_{00}, L_{01}, L_{11} \in k[x_2, \ldots, x_n].$$

Our assumption that X is smooth means that the system

$$\partial F/\partial x_0 | \ell = \ldots = \partial F/\partial x_n | \ell = 0$$

has no nontrivial solutions. The derivatives $\partial F/\partial x_0$ and $\partial F/\partial x_1$ are zero along ℓ. Using our expansion for F, we are left with the system

$$A_j := x_0^2 \partial L_{00}/\partial x_j + x_0 x_1 \partial L_{01}/\partial x_j + x_1^2 \partial L_{11}/\partial x_j = 0, \quad j = 2, \ldots, n.$$

These are binary quadratic forms in x_0, x_1.

Two quadrics $A_i, A_j \in k[x_0, x_1]$ have a common factor if and only if

$$\langle A_i, A_j \rangle_3 = k[x_0, x_1]_3.$$

This follows from the construction of the resultant on two quadrics. Thus if $\langle A_j, j = 2, \ldots, n \rangle \subset k[x_0, x_1]$ is an ideal generated by quadratic forms then $\langle A_j \rangle$ is irrelevant if and only if

$$\langle A_j \rangle_3 = k[x_0, x_1]_3. \tag{1}$$

We apply Theorem 10. We have

$$T_{[\ell]} F_1(X) = \{\phi : \phi(p) \in \mathbb{T}_p X \text{ for each } p \in X\}.$$

Represent ϕ by the transpose B^t where

$$B = \begin{pmatrix} b_{02} & b_{03} & \dots & b_{0n} \\ b_{12} & b_{13} & \dots & b_{1n} \end{pmatrix}.$$

Thus we find

$$T_{[\ell]}F_1(X) = \{B : \sum_{j=2}^{n} \partial F/\partial x_j|\ell \cdot (x_0 b_{0j} + x_1 b_{1j}) = 0$$

which is equal to

$$\{B : \sum_{j=2}^{n} A_j(x_0 b_{0j} + x_1 b_{1j}) = 0\},$$

i.e., the linear syzygies among the A_j. In other words, our tangent space is the kernel of the linear transformation

$$\begin{array}{rcl} k[x_0, x_1]_1^{\oplus n-1} & \rightarrow & k[x_0, x_1]_3 \\ (B_j = b_{0j}x_0 + b_{1j}x_1)_{j=2,\dots,n} & \mapsto & \sum_{j=2}^{n} B_j A_j. \end{array}$$

Equation 1 says this is surjective, so the kernel has dimension

$$2(n-1) - 4 = 2(n-3),$$

the expected dimension of $F_1(X)$. $\square$

Exercise 14 Give an example of a smooth cubic hypersurface $X \subset \mathbb{P}^n$ where $F_m(X), m > 1$ fails to be smooth of the expected dimension. Can you find an example where $F_m(X)$ is of the expected dimension but fails to be smooth?

Exercise 15 Let $X = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0\} \subset \mathbb{P}^4$ be the Fermat quartic threefold. Show that $F_1(X)$ is singular.

3.4 Lines on cubic surfaces

Theorem 13 implies that the variety (scheme) of lines on a smooth cubic surface is smooth of dimension zero. We refine this result

Proposition 16 *Each smooth cubic surface contains exactly* 27 *lines.*

The first step in proving this is to show that *some* smooth cubic surface contains 27 lines.

Exercise 17 (see [6, V.4]) Choose $p_1, \ldots, p_6 \in \mathbb{P}^2$ distinct points in general position, i.e., no three points collinear and all six not contained in a conic. Show that

1. The linear series of homogeneous cubics vanishing at $p_1, \ldots, p_6$ has projective dimension three. It has no base points outside $p_1, \ldots, p_6$.
2. Consider the blow-up
$$\beta : X = \mathrm{Bl}_{p_1,\ldots,p_6}\mathbb{P}^2 \to \mathbb{P}^2$$
with exceptional divisors $E_1, \ldots, E_6$. Write L for the pull-back of the hyperplane class from $\mathbb{P}^2$. Show that the linear series $|3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6|$ induces a morphism
$$j : X \to \mathbb{P}^3.$$
Hint: Relate this to the linear series in the previous part.
3. Verify that j is an embedding and its image is a cubic surface.
4. Show that the exceptional curves $E_i \subset X$ appear as lines on $X \subset \mathbb{P}^3$. Verify that the proper transforms of lines through two of the points (with divisor class $L - E_i - E_j$) and conics through five of the points (with divisor class $2L - E_i - E - j - E_k - E_l - E_m$) are also lines on X.
5. **Challenge:** Show that any line on X is one of the 27 enumerated above.

The difficulty in proving Proposition 16 is that we do not know *a priori* that every smooth cubic surface arises from the blow-up construction. To establish this, we will need some general results for hypersurfaces.

Proposition 18 *Consider the incidence correspondence*
$$\mathcal{Z} \subset \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_d) \times \mathbb{G}(m, n)$$
defined by
$$\{([F], \Lambda) : F|\Lambda = 0\}.$$
The projection $\pi_1 : \mathcal{Z} \to \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_d)$ *is a projective morphism.*

proof: The Grassmannian $\mathbb{G}(m,n)$ is a projective variety; this can be found in any book discussing the Plücker embedding (e.g., [7, ch. 11]). The projection π_2 is a projective bundle of relative dimension

$$\dim \mathbb{P}(\mathbb{C}[x_0,\dots,x_n]_d) - \binom{d+m}{d} = \binom{d+n}{d} - \binom{d+m}{d} - 1,$$

so the total space $\mathcal{Z}$ is projective. Thus the projection π_1 is projective as well. □

Proposition 19 *Consider the locus*

$$U \subset \mathbb{P}(\mathbb{C}[x_0,\dots,x_n]_d)$$

parametrizing smooth hypersurfaces. Then U is Zariski open and is a connected manifold.

proof: Consider the incidence correspondence

$$\mathcal{W} \subset \mathbb{P}^n \times \mathbb{P}(\mathbb{C}[x_0,\dots,x_n]_d)$$

describing pairs

$$\{(p,[F]) : p \in \operatorname{Sing}(X(F))\}.$$

Let ψ_1 and ψ_2 denote the projection morphisms. We claim $\mathcal{W}$ is closed. There are two ways to see this. The first relies on the geometry of ψ_1. The set of all hypersurfaces of degree d with a singularity at p forms a codimension-$(n+1)$ subspace of $\mathbb{P}(\mathbb{C}[x_0,\dots,x_n])$; indeed, if $p=[0,0,\dots,0,1]$ then we are insisting that the coefficients of the monomials

$$\{x_n^d, x_n^{d-1}x_0, \dots, x_n^{d-1}x_{n-1}\}$$

all vanish. Thus $\mathcal{W}$ is a projective bundle over $\mathbb{P}^n$ and hence is closed of dimension

$$\binom{n+d}{d} - 2.$$

The second uses the explicit description of the bihomogeneous equations of $\mathcal{W}$

$$\mathcal{W} = X(\langle \partial F/\partial x_i,\ i=0,\dots,n\rangle),$$

where the coefficients of F are identified with the coordinates on $\mathbb{P}(\mathbb{C}[x_0,\dots,x_n]_d)$.

Since $\mathcal{W}$ is closed its image

$$\psi_2(\mathcal{W}) \subset \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_d)$$

is closed. Since U is the complement of $\psi_2(\mathcal{W})$, it is necessarily open.

To see that U is connected, it suffices to verify it is pathwise connected. Given $p, q \in U$, consider the line

$$p, q \in \ell(p, q) \subset \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_d).$$

The intersection $\ell(p, q) \cap U$ is Zariski open in $\ell(p, q) \simeq \mathbb{P}^1$, i.e., it has finite complement. We find that

$$\ell(p, q) \cap U = \text{Riemann sphere} \setminus \text{finite set}$$

which is pathwise connected.$\square$

Exercise 20 Show that

$$\psi_2(\mathcal{W}) \subset \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_d)$$

is a nonempty divisor, called the *discriminant.*

Exercise 21 Compute an explicit equation for the discriminant when $d = 2$. When $d = 3$ and $n = 2$ show that the discriminant has degree 12. When $d = 3$ and $n = 3$ verify the discriminant has degree 32.

proof of Proposition 16: Here the incidence correspondence takes the form

$$\mathcal{Z} \subset \mathbb{P}(\mathbb{C}[x_0, x_1, x_2, x_3]_3) \times \mathbb{G}(1, 3) = \mathbb{P}^{19} \times \mathbb{G}(1, 3).$$

Consider the restriction to the smooth cubic surfaces

$$\pi_1 : \mathcal{Z}_U = \pi_1^{-1}(U) \to U.$$

This morphism is quasi-finite and projective, hence finite. It is unramified by Theorem 13, and hence is a covering space in the sense of algebraic topology. Proposition 19 implies U is connected, so the number of lines on a smooth cubic surface is constant. Exercise 17 shows this number is 27. $\square$

Remark 22 Singular cubics can have lots of lines:

Exercise 23 For $X = \{x_0x_1x_2 = 0\} \subset \mathbb{P}^3$, show that $F_1(X)$ consists of a union of three copies of $\mathbb{P}^2$. Describe how these intersect.

For $X = \{x_0^3 + x_1^3 + x_2^3 = 0\} \subset \mathbb{P}^3$, show that $F_1(X)$ is nonreduced but supported along a smooth curve.

Exercise 24 Let X be a cubic surface singular at $p = [1, 0, 0, 0]$. Verify that the defining equation for X can be expressed

$$F = x_0G_2(x_1, x_2, x_3) + G_3(x_1, x_2, x_3) = 0,$$

where G_2 and G_3 are homogeneous. Show that the generic such X contains six lines passing through p, each with multiplicity two in $F_1(X)$.

Exercise 25 Enumerate the lines in the cubic surfaces

$$x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_3x_0x_1 = 0$$

and

$$x_0x_1x_2 = x_3^3$$

and compute the multiplicity of each line. *Hint:* If there are a finite number of lines, the sum of the multiplicities must be 27.

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