Math 567 Lecture Notes

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1 Why rational curves?

Before answering this rhetorical question, we should give a working definition for our object of study:

Definition 1 Let X be a smooth projective variety over $\mathbb{C}$. A *rational curve* on X is the image of a nonconstant morphism

$$f : \mathbb{P}^1 \to X.$$

Throughout, a 'morphism' is a morphism of algebraic varieties (or schemes), unless otherwise specified.

The rational curves on a variety govern its place in the classification of all varieties. Varieties with lots of rational curves are qualitatively different from varieties without rational curves in fundamental ways.

1.1 Elementary Examples

$X = \mathbb{P}^n$:
It is an elementary fact that there exists a rational curve through any pair of points $p, q \in \mathbb{P}^n$, e.g., the line $\ell(p, q)$ determined by these points. Writing

$$p = [p_0, \ldots, p_n], \quad q = [q_0, \ldots, q_n]$$

we have a morphism

$$\begin{array}{rcl} f : \mathbb{P}^1 & \to & \mathbb{P}^n \\ [s, t] & \mapsto & [sp_0 + tq_0, \ldots, sp_n + tq_n]. \end{array}$$

Exercise 2 Given $p_1, \ldots, p_N \in \mathbb{P}^n$, show there exists a rational curve containing these points.

We sketch an approach to this problem. The main ingredient is the following classical fact [4, pp.14]:

> Fix points $q_1, \ldots, q_{d+3} \in \mathbb{P}^d$ in linear general position, i.e., no $d+1$ of the points lie in a hyperplane. Then there exists a unique rational normal curve of degree d through these points.

Then realize $p_1, \ldots, p_N$ as the linear projection of a collection of points in linear general position lying on a rational normal curve. Take the projection of that curve as the desired rational curve.

Quadric hypersurface $X \subset \mathbb{P}^n$, $(n \geq 3)$:
There exists a linear change of coordinates such that

$$X = \{[x_0, \ldots, x_n] : x_0^2 + \ldots + x_n^2 = 0\}.$$

Exercise 3 Given $x_1, \ldots, x_N \in X$, show there exists a rational curve $R \subset X$ containing these points.

One strategy is to reduce this to Exercise 2. Choose a point $x_0 \in X$ such that the tangent hyperplane to X at x_0 contains none of the $x_1, \ldots, x_N$. Consider projection from x_0

$$\pi_{x_0} : X \overset{\sim}{\dashrightarrow} \mathbb{P}^{n-1}$$

which maps X birationally to $\mathbb{P}^{n-1}$. Find a rational curve in $\mathbb{P}^{n-1}$ containing the $\pi_{x_0}(x_1), \ldots, \pi_{x_0}(x_N)$.

1.2 Less elementary examples

Riemann surface X of genus $g > 0$:

Exercise 4 Any morphism

$$f : \mathbb{P}^1 \to X$$

is constant.

Hint: Apply the Hurwitz formula.

$X = \mathbb{C}^g/\Lambda$ complex torus:

Exercise 5 Any morphism

$$f : \mathbb{P}^1 \to X$$

is constant.

Note that $\mathbb{C}^g \to X$ is the universal covering space of X. Since $\mathbb{P}^1$ is simply connected, there exists a lift h of f

$$\begin{array}{ccc} & h & \mathbb{C}^g \\ & \nearrow & \downarrow \\ \mathbb{P}^1 & \xrightarrow{f} & X. \end{array}$$

Indeed, any map from a simply-connected space lifts to the universal covering. The lift h is holomorphic. Since any holomorphic map from $\mathbb{P}^1$ to $\mathbb{C}$ is constant, h and hence f is constant.

1.3 Harder examples

$X \subset \mathbb{P}^n$ hypersurfaces of degree $d \geq 2n - 2$:

We explain what the *generic hypersurface* means in this context: Let $\mathbb{P}^{\binom{d+n}{n}-1}$ denote the parameter space for hypersurfaces of degree d in $\mathbb{P}^n$. We say that a property holds for a generic hypersurface if there exists a nonempty Zariski open subset of $\mathbb{P}^{\binom{d+n}{n}-1}$ over which the property holds.

Theorem 6 *[2] [6] A generic hypersurface $X \subset \mathbb{P}^n$ of degree d contains* no *rational curves provided $d \geq 2n - 1$ or $d \geq 2n - 2$ with $n \geq 4$.*

For specific hypersurfaces there might be rational curves, e.g., for the Fermat surface

$$X = \{x_0^5 + x_1^5 = x_2^5 + x_3^5\} \subset \mathbb{P}^3$$

we have the line

$$\ell = \{x_0 - x_2 = x_1 - x_3 = 0\}.$$

Conjecture 7 (Lang conjecture [5]) Let $X \subset \mathbb{P}^3$ be a smooth surface of degree ≥ 5. Then there a finite number of rational curves on X.

Quartic hypersurfaces $X \subset \mathbb{P}^3$:

The case $n = 3, d = 4$ is a curious omission in Theorem 6. The following assertion is attributed to Segre [3], but was proven by S. Nakatani and Xi Chen [1]

Theorem 8 *A general quartic surface contains an infinite number of rational curves.*

By definition, a property holds for a *general hypersurface* if there exists a countable intersection of nonempty Zariski open subsets of $\mathbb{P}^{\binom{d+n}{n}-1}$ (the parameter space for all hypersurfaces) over which the property holds.

Conjecture 9 Each smooth quartic surface contains an infinite number of rational curves.

References

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