Math 468 Lecture Notes

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The goal today is to illustrate how the theory of induced representations can be used to construct representations fairly explicitly. The example we shall focus on is $\mathfrak{S}_5$, the symmetric group on five letters.

Preliminaries

We first recall a couple facts from previous lectures:

1. Suppose we are given a representation $\rho : G \to \mathrm{GL}(V)$, and W is an irreducible representation of G appearing in V with multiplicity *one*. Then there is an canonical formula for the projection map
$$p : V \to W.$$
2. Let $H \subset G$ be a subgroup, R a collection of coset representatives of G/H, $\theta : H \to \mathrm{GL}(W)$ a representation, and $\rho : G \to \mathrm{GL}(V)$ a representation induced from θ. Then we can write
$$\chi_\rho(u) = \sum_{r \in R : r^{-1}ur \in H} \chi_\theta(r^{-1}ur). \tag{1}$$

Review of conjugacy classes of $\mathfrak{S}_d$ Let $\sigma \in \mathfrak{S}_d$ denote a permutation. Recall that σ can be written uniquely as a product of disjoint cycles, e.g.,
$$\sigma = (1)(23)(78)(456) \in \mathfrak{S}_9.$$
This reflects the decomposition of $\{1, \ldots, d\}$ into orbits under the action of $\langle\sigma\rangle$, the subgroup generated by σ. Let $\mu_1, \mu_2, \ldots, \mu_k$, denote the lengths of

the cycles that appear in σ, ordered so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq 1$. Since $\mu_1 + \ldots + \mu_k = d$, the collection $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ encodes a *partition* of d. We write $|\mu| = \sum_j \mu_j$. So for the permutation σ above, we have $\mu = (3, 2, 2, 1)$ corresponding to the partition $8 = 3 + 2 + 2 + 1$.

Here is another set of notation for these decompositions: For each $j = 1, \ldots, d$ let

$$i_j = \#\{k : \mu_k = j\},$$

i.e., i_j is the number of j-cycles. Then we have $d = \sum_{j=1}^{d} j i_j$.

Thus we have identifications

$$\{\text{conjugacy classes of } \mathfrak{S}_d\} \Leftrightarrow \{\text{partitions } \mu \text{ of } d\} \Leftrightarrow$$
$$\{\text{sequences of nonnegative integers } i = (i_1, \ldots, i_d) : \textstyle\sum_{j=1}^{d} i_j = d\}$$

Proposition 0.1 *Let $C_i \subset \mathfrak{S}_d$ denote the conjugacy class associated with i. Then we have*

$$|C_i| = \frac{d!}{1^{i_1} i_1! 2^{i_2} i_2! \cdots d^{i_d} i_d!}.$$

Proof: Write $k = i_1 + \cdots + i_d$; express each permutation in C_i is a product

$$\sigma = c_1 \cdots c_k$$

where $c_j = (c_{j1} \cdots c_{j\mu_j})$ is a cycle of length μ_j. We can regard the sequence of integers

$$c_{11}, \ldots c_{1\mu_1}, c_{21}, \ldots, c_{k\mu_k}$$

as a permutation of $\{1, \ldots, d\}$.

In total there are $d!$ such permutations. However, distinct permutations can give rise to same element σ. First, the cycles $(c_{j1} c_{j2} \cdots c_{j\mu_j})$ and $(c_{j2} \cdots c_{j\mu_j} c_{j1})$ are equivalent; this means we have to divide out by

$$\mu_1 \cdots \mu_k 1^{i_1} \cdots d^{i_d}.$$

Second, we could permute the i_j cycles of length j to get an equivalent permutation; this involves dividing out by

$$i_1! i_2! \cdots i_d!.$$

These account for the formula in the proposition. $\square$

Special subgroups in $\mathfrak{S}_d$ Here we enumerate special conjugacy classes of subgroups of $\mathfrak{S}_d$ associated with partitions λ of d.

Definition 0.2 *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition of d. We define the* Young subgroup

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_\ell} \subset \mathfrak{S}_d,$$

where $\mathfrak{S}_{\lambda_j}$ is regarded as the permutations of the integers

$$\{\lambda_1 + \cdots + \lambda_{j-1} + 1, \ldots, \lambda_1 + \cdots + \lambda_j\}.$$

We have

$$|\mathfrak{S}_\lambda| = \lambda_1!\lambda_2! \cdots \lambda_\ell!.$$

Of course, reordering $\{1, \ldots, d\}$ gives us another version of the same subgroup.

For example, $\mathfrak{S}_{(2,2)} \subset \mathfrak{S}_4$ is the group $\langle (12), (34) \rangle$ of order four. Or we could take the conjugate subgroup $\langle (13), (24) \rangle$.

Definition 0.3 *For each partition λ of d, let U_λ denote the representation of $\mathfrak{S}_d$ induced from the* trivial *representation of $\mathfrak{S}_\lambda$. Let υ_λ denote its character.*

Thus U_λ is a permutation representation associated with the coset representation of $\mathfrak{S}_d$ on $\mathfrak{S}_d/\mathfrak{S}_\lambda$. Its matrices have entries equal to 0 or 1.

Analysis of $\mathfrak{S}_5$

Conjugacy classes of $\mathfrak{S}_5$ We first tabulate the number of elements in each conjugacy class:

	(12345)	(1234)	(123)(45)	(123)	(12)(34)	(12)	identity
μ	(5)	$(4,1)$	$(3,2)$	$(3,1,1)$	$(2,2,1)$	$(2,1,1,1)$	$(1,1,1,1,1)$
i	$(0,0,0,0,1)$	$(1,0,0,1,0)$	$(0,1,1,0,0)$	$(2,0,1,0,0)$	$(1,2,0,0,0)$	$(3,1,0,0,0)$	$(5,0,0,0,0)$
$\|C_i\|$	24	30	20	20	15	10	1

Trivial representation Since $\mathfrak{S}_{(5)} = \mathfrak{S}_5$, the representation $U_{(5)}$ is the trivial representation of $\mathfrak{S}_5$. We also use the notation $V_{(5)}$ for this, and $\chi_{(5)}$ for the associated character.

Standard representation Consider $\mathfrak{S}_{(4,1)} = \mathfrak{S}_4 \times \mathfrak{S}_1 = \mathfrak{S}_4$ and the representation $U_{(4,1)}$. Note $\mathfrak{S}_5/\mathfrak{S}_4$ has five cosets, and the associated coset representation is the standard permutation representation on five letters. Thus $U_{(4,1)}$ is the permutation representation

$$\rho_\sigma \cdot e_j = e_{\sigma(j)}, \quad j = 1, \ldots, 5.$$

We've seen that this decomposes

$$U_{(4,1)} = \text{irred.} \oplus \text{trivial} = V_{(4,1)} \oplus V_{(5)},$$

where $V_{(4,1)}$ is the nontrivial summand. Let $\chi_{(4,1)}$ denote its character.

Furthermore, in an exercise from chapter two of Serre, we computed the character in terms of the number of fixed elements:

μ	(5)	(4, 1)	(3, 2)	(3, 1, 1)	(2, 2, 1)	(2, 1, 1, 1)	(1, 1, 1, 1, 1)
$\upsilon_{(4,1)}$	0	1	0	2	1	3	5

Subtracting the trivial character, we obtain

μ	(5)	(4, 1)	(3, 2)	(3, 1, 1)	(2, 2, 1)	(2, 1, 1, 1)	(1, 1, 1, 1, 1)
$\chi_{(4,1)}$	−1	0	−1	1	0	2	4

The representation $U_{(3,2)}$ The group $\mathfrak{S}_{(3,2)} = \mathfrak{S}_3 \times \mathfrak{S}_2$ has order twelve; thus the induced representation $U_{(3,2)}$ has dimension ten.

We compute the character $\upsilon_{(3,2)}$ using formula (1). This can be interpretted combinatorially: To compute $\upsilon_{(3,2)}(C_\mu)$, we must count the number of ways the partition μ can be separated into subpartitions with 3 and 2 elements, modulo the action of $\mathfrak{S}_3 \times \mathfrak{S}_2$. For $\mu = (5)$ and $(4,1)$ there no such subpartitions. The $\mu = (3,2)$ and $(3,1,1)$ there is a unique such partition. For $\mu = (2,2,1)$ there are two subparitions, i.e., $5 = \{2+1\} + \{2'\} = \{2'+1\} + \{2\}$. For $\mu = (2,1,1,1)$ we have four subpartitions:

$$5 = \{2+1\}+\{1'+1''\} = \{2+1'\}+\{1+1''\} = \{2+1''\} = \{1+1'\} = \{1+1'+1''\}+\{2\};$$

for $\mu = (1,1,1,1,1)$ we have ten.

Thus we have:

μ	(5)	(4, 1)	(3, 2)	(3, 1, 1)	(2, 2, 1)	(2, 1, 1, 1)	(1, 1, 1, 1, 1)
$\upsilon_{(3,2)}$	0	0	1	1	2	4	10

We compute inner products with irreducible characters we already have:

$$\left(\upsilon_{(3,2)}|\chi_{(5)}\right) = \frac{1}{120}(20+20+15(2)+10(4)+10) = 1$$

and

$$\left(\upsilon_{(3,2)}|\chi_{(4,1)}\right) = \frac{1}{120}(-20+20+80+40) = 1.$$

Write

$$U_{(3,2)} = V_{(3,2)} \oplus V_{(4,1)} \oplus V_{(5)}$$

where $V_{(3,2)}$ consists of the irreducible components distinct from $V_{(4,1)}$ and $V_{(5)}$. Let $\chi_{(3,2)}$ be the character of the first summand:

μ	(5)	(4, 1)	(3, 2)	(3, 1, 1)	(2, 2, 1)	(2, 1, 1, 1)	(1, 1, 1, 1, 1)
$\chi_{(3,2)}$	0	−1	1	−1	1	1	5

Since

$$\left(\chi_{(3,2)}|\chi_{(3,2)}\right) = \frac{1}{120}(30+20+20+15+10+25) = 1$$

we conclude $V_{(3,2)}$ is irreducible.

The representation $U_{(3,1,1)}$ The group $\mathfrak{S}_{(3,1,1)}$ has order six, so $\dim U_{(3,1,1)} = 20$. Arguing similarly to the previous case, we find:

μ	(5)	(4, 1)	(3, 2)	(3, 1, 1)	(2, 2, 1)	(2, 1, 1, 1)	(1, 1, 1, 1, 1)
$\upsilon_{(3,1,1)}$	0	0	0	2	0	6	20

Computing inner products of characters, we obtain

$$\left(\upsilon_{(3,1,1)}|\chi_{(5)}\right) = 1, \left(\upsilon_{(3,1,1)}|\chi_{(4,1)}\right) = 2, \left(\upsilon_{(3,1,1)}|\chi_{(3,2)}\right) = 1.$$

Thus we find

$$U_{(3,1,1)} = V_{(3,1,1)} \oplus V_{(3,2)} \oplus V_{(4,1)}^{\oplus 2} \oplus V_{(5)},$$

where $V_{(3,1,1)}$ contains the components distinct from the last three irreducibles. If $\chi_{(3,1,1)}$ denote its character we find

μ	(5)	(4, 1)	(3, 2)	(3, 1, 1)	(2, 2, 1)	(2, 1, 1, 1)	(1, 1, 1, 1, 1)
$\chi_{(3,1,1)}$	1	0	0	0	−2	0	6

and

$$\left(\chi_{(3,1,1)}, \chi_{(3,1,1)}\right) = 1.$$

Thus $V_{(3,1,1)}$ is irreducible as well.

The representation $U_{(2,2,1)}$ The group $\mathfrak{S}_{(2,2,1)}$ has order four, so $\dim U_{(2,2,1)} = 30$. Enumerating subpartitions as above, we find:

μ	(5)	$(4,1)$	$(3,2)$	$(3,1,1)$	$(2,2,1)$	$(2,1,1,1)$	$(1,1,1,1,1)$
$\upsilon_{(2,2,1)}$	0	0	0	0	2	6	30

The inner products with the irreducibles we have are:

$\big(\upsilon_{(2,2,1)}|\chi_{(5)}\big) = 1$, $\big(\upsilon_{(2,2,1)}|\chi_{(4,1)}\big) = 2$, $\big(\upsilon_{(2,2,1)}|\chi_{(3,2)}\big) = 2$, $\big(\upsilon_{(2,2,1)}|\chi_{(3,1,1)}\big) = 1$.

Thus we find

$$U_{(2,2,1)} = V_{2,2,1} \oplus V_{(3,1,1)} \oplus V_{(3,2)}^{\oplus 2} \oplus V_{(4,1)}^{\oplus 2} \oplus V_{(5)},$$

where $V_{(2,2,1)}$ contains the components distinct from the four known irreducibles. If $\chi_{(2,2,1)}$ denotes its character we have

μ	(5)	$(4,1)$	$(3,2)$	$(3,1,1)$	$(2,2,1)$	$(2,1,1,1)$	$(1,1,1,1,1)$
$\chi_{(2,2,1)}$	0	1	-1	-1	1	-1	5

and

$$\big(\chi_{(2,2,1)}, \chi_{(2,2,1)}\big) = 1.$$

Thus $V_{(2,2,1)}$ is irreducible as well.

The representation $U_{(2,1,1,1)}$ The group $\mathfrak{S}_{(2,1,1,1)}$ has order two, thus $\dim U_{(2,1,1,1)} = 60$. Computing as above, we find:

μ	(5)	$(4,1)$	$(3,2)$	$(3,1,1)$	$(2,2,1)$	$(2,1,1,1)$	$(1,1,1,1,1)$
$\upsilon_{(2,1,1,1)}$	0	0	0	0	0	6	60

The inner products with the irreducibles we have are:

$$\big(\upsilon_{(2,1,1,1)}|\chi_{(5)}\big) = 1, \big(\upsilon_{(2,1,1,1)}|\chi_{(4,1)}\big) = 3, \big(\upsilon_{(2,1,1,1)}|\chi_{(3,2)}\big) = 3,$$
$$\big(\upsilon_{(2,1,1,1)}|\chi_{(3,1,1)}\big) = 3, \big(\upsilon_{(2,1,1,1)}|\chi_{(2,2,1)}\big) = 2.$$

Thus we find

$$\begin{aligned} U_{(2,1,1,1)} &= V_{2,1,1,1} \oplus V_{2,2,2}^{\oplus 2} \oplus V_{(3,1,1)}^{\oplus 3} \\ &\quad \oplus V_{(3,2)}^{\oplus 3} \oplus V_{(4,1)}^{\oplus 3} \oplus V_{(5)}, \end{aligned}$$

where $V_{(2,1,1,1)}$ consists of the summands distinct from the known irreducibles. If $\chi_{(2,1,1,1)}$ denotes its character we have

μ	(5)	$(4,1)$	$(3,2)$	$(3,1,1)$	$(2,2,1)$	$(2,1,1,1)$	$(1,1,1,1,1)$
$\chi_{(2,1,1,1)}$	-1	0	1	1	0	-2	4

and

$$\left(\chi_{(2,1,1,1)}, \chi_{(2,1,1,1)}\right) = 1.$$

Thus $V_{(2,1,1,1)}$ is also irreducible.

The representation $U_{(1,1,1,1,1)}$ Here we are inducing from the trivial representation, so $U_{(1,1,1,1,1)}$ is the regular representation. Thus we have

μ	(5)	(4, 1)	(3, 2)	(3, 1, 1)	(2, 2, 1)	(2, 1, 1, 1)	(1, 1, 1, 1, 1)
$\upsilon_{(1,1,1,1,1)}$	0	0	0	0	0	0	120

We know that the regular representations decomposes

$$U_{(1,1,1,1,1)} = V_{1,1,1,1,1} \oplus V_{2,1,1,1}^{\oplus 4} \oplus V_{2,2,2}^{\oplus 5} \oplus V_{(3,1,1)}^{\oplus 6} \oplus V_{(3,2)}^{\oplus 5} \oplus V_{(4,1)}^{\oplus 4} \oplus V_{(5)},$$

where $V_{1,1,1,1,1}$ is the summand containing the final irreducible. Its character $\chi_{(1,1,1,1,1)}$ is

μ	(5)	(4, 1)	(3, 2)	(3, 1, 1)	(2, 2, 1)	(2, 1, 1, 1)	(1, 1, 1, 1, 1)
$\chi_{(1,1,1,1,1)}$	1	−1	−1	1	1	−1	1

Of course, we recognize this as the character of the alternating representation.

We summarize these computations:

Proposition 0.4 *Consider the symmetric group $\mathfrak{S}_5$, and the Young subgroups $\mathfrak{S}_\lambda$ for each partition λ. Let U_λ denote the representation on $\mathfrak{S}_5$ induced from the trivial representation of $\mathfrak{S}_\lambda$. Then U_μ admits a distinguished irreducible representation V_λ, and every irreducible representation of $\mathfrak{S}_5$ arises in this way.*

Corollary 0.5 There is a *natural* bijection between conjugacy classes $C_\lambda \subset \mathfrak{S}_5$ and irreducible representations V_λ of $\mathfrak{S}_5$.

Corollary 0.6 All characters of $\mathfrak{S}_5$ take values in the integers. Every representation can be written using matrices with rational entries.

We prove the last statement: The matrices appearing in U_λ have integer entries. The formula for the projection

$$p_\lambda : U_\lambda \to V_\lambda$$

was computed explicitly in terms of χ_λ; see chapter two of Serre. The formula there implies that the entries of matrices arising in V_λ have denominator dividing 5!.

Tabulation of the characters of U_μ

μ	(5)	(4, 1)	(3, 2)	(3, 1, 1)	(2, 2, 1)	(2, 1, 1, 1)	(1, 1, 1, 1, 1)
$\upsilon_{(5)}$	1	1	1	1	1	1	1
$\upsilon_{(4,1)}$	0	1	0	2	1	3	5
$\upsilon_{(3,2)}$	0	0	1	1	2	4	10
$\upsilon_{(3,1,1)}$	0	0	0	2	0	6	20
$\upsilon_{(2,2,1)}$	0	0	0	0	2	6	30
$\upsilon_{(2,1,1,1)}$	0	0	0	0	0	6	60
$\upsilon_{(1,1,1,1,1)}$	0	0	0	0	0	0	120

Tabulation of the irreducible characters

μ	(5)	(4, 1)	(3, 2)	(3, 1, 1)	(2, 2, 1)	(2, 1, 1, 1)	(1, 1, 1, 1, 1)
$\chi_{(5)}$	1	1	1	1	1	1	1
$\chi_{(4,1)}$	−1	0	−1	1	0	2	4
$\chi_{(3,2)}$	0	−1	1	−1	1	1	5
$\chi_{(3,1,1)}$	1	0	0	0	−2	0	6
$\chi_{(2,2,1)}$	0	1	−1	−1	1	−1	5
$\chi_{(2,1,1,1)}$	−1	0	1	1	0	−2	4
$\chi_{(1,1,1,1,1)}$	1	−1	−1	1	1	−1	1

Reference: Fulton and Harris, *Representation Theory: A First Course*, Springer-Verlag, 1996