# Math 468 Lecture Notes

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**Application of invariant inner products** We start with some basic linear algebra

**Proposition 0.1** *Let* $(|)$ *be an inner product on* $\mathbb{R}^n$*. Then there exists a basis* $f_1, \ldots, f_n$ *of* $\mathbb{R}^n$ *such that*

$$(v_1 f_1 + \cdots + v_n f_n | w_1 f_1 + \cdots + w_n f_n) = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

In other words, each inner product can be transformed to the standard inner product by a suitable linear change of coordinates.

*Proof:* Let $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{R}^n$; let $A$ denote the symmetric matrix associated to $(|)$, i.e.,

$$(v|w) = v^t A w.$$

We can compute the entries $a_{ij}$ of $A$ by the formula

$$a_{ii} = (e_i|e_i), \quad a_{ij} = (e_i|e_j)/2, i \neq j.$$

The Spectral Theorem implies the existence of an orthonormal basis $e'_1, \ldots, e'_n$ over which $A$ is diagonal with real eigenvalues $\lambda_1, \ldots, \lambda_n$, i.e.,

$$\left(e'_i|e'_j\right) = \lambda_i \delta_{ij}.$$

Since the inner product is positive definite, each eigenvalue is positive. Take $f_i = e'_i/\sqrt{\lambda_i}$, so that

$$(f_i|f_j) = \delta_{ij}.$$

**Corollary 0.2** Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a finite group, where $V$ is a real vector space of dimension $n$. Then $\rho$ is conjugate to representation $\rho' : G \to \mathrm{O}(\mathbb{R}^n)$, where $\mathrm{O}(\mathbb{R}^n)$ is the group of orthogonal matrices with respect to the standard inner product on $\mathbb{R}^n$.

*Proof:* Let $(|)$ be an inner product on $V$ invariant under the action of $G$. Applying the proposition, we obtain a basis for $V$ for which this is equal to the standard inner product. When we write $\rho$ in terms of this basis, the elements $\rho(\sigma), \sigma \in G$, preserve the standard inner product, and thus are orthogonal.

Here we give a concrete application: Let $\rho : G \hookrightarrow \mathrm{GL}(\mathbb{R}^2)$ denote an injective representation of a finite group. The corollary shows that this is conjugate to a representation

$$\rho' : G \hookrightarrow \mathrm{O}(\mathbb{R}^2).$$

Recall that elements of the orthogonal group have determinant $\pm 1$. Define

$$H = \{\sigma \in G : \det \rho'(\sigma) = 1\} \subset G,$$

which is either equal to $G$ (if every element there has determinant one) or has index two.

Now the subgroup

$$\mathrm{SO}(\mathbb{R}^2) = \{A \in \mathrm{O}(\mathbb{R}^2) : \det A = 1\} = \{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\},$$

the group of rotations of the plane. This is an abelian group and every finite subgroup is cyclic; in particular, $H$ is the set of rotations by $2\pi j/n$ radians, for some $n \in \mathbb{N}$.

If $G = H$ then $G$ is also a group of rotations. Otherwise, $G$ contains an element of determinant $-1$, i.e., a reflection, which is not contained in $H$. In this case, $G = D_n$ the dihedral group of order $2n$. Choosing the coordinate axes in such a way that the reflection fixes the $x$-axis, we recover the representation for $D_n$ introduced in the first class.

To summarize:

**Proposition 0.3** *Any finite subgroup of* $\mathrm{GL}(\mathbb{R}^2)$ *is isomorphic to* $C_n$ *and* $D_n$*, and the representation is conjugate to the representations introduced previously, i.e., if* $C_n = \langle \sigma : \sigma^n = 1 \rangle$ *and* $D_n = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^2 \rangle$

*then*

$$\begin{aligned} \sigma &\mapsto \begin{pmatrix} \cos 2\pi/n & -\sin 2\pi/n \\ \sin 2\pi/n & \cos 2\pi/n \end{pmatrix} \\ \tau &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

**Decomposition of a regular representation into irreducibles** Let $\rho : C_n \to \mathrm{GL}(\mathbb{C}^n)$ denote the regular representation of $C_n = \langle \sigma : \sigma^n = 1 \rangle$. We write $\{e_1, e_\sigma, \ldots, e_{\sigma^{n-1}}\}$ for the standard basis such that

$$\rho(\sigma) \cdot e_{\sigma^j} = e_{\sigma^{j+1}}.$$

To decompose this into irreducibles, the key step is to diagonalize the generator $\rho(\sigma)$. How do we exhibit eigenvectors of this?

We have already seen that averaging the basis vectors of a regular representation gives an invariant vector

$$v_0 = e_1 + e_\sigma + \cdots + e_{\sigma^{n-1}},$$

i.e., $\rho(\sigma) \cdot v_0 = v_0$. Let $\zeta = e^{2\pi i/n}$ be a primitive $n$th root of unity. Consider the 'weighted' average

$$v_1 = e_1 + \zeta e_\sigma + \cdots + \zeta^{n-1} e_{\sigma^{n-1}} = \sum_{i=0}^{n-1} \zeta^i e_{\sigma^i},$$

which satisfies

$$\rho(\sigma) \cdot v_1 = \zeta^{-1} v_1.$$

More generally, for each $j = 0, \ldots, n-1$ we can consider

$$v_j = \sum_{i=0}^{n-1} \zeta^{ij} e_{\sigma^i},$$

which satisfies

$$\rho(\sigma) \cdot v_j = \zeta^{-j} v_j.$$

We claim that $\{v_0, \ldots, v_{n-1}\}$ is an eigenbasis for $\mathbb{C}^n$ under the action of $C_n$. To see these are linear independent, consider the change-of-basis matrix

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta & \zeta^2 & \cdots & \zeta^{n-1} \\ 1 & \zeta^2 & \zeta^4 & \cdots & \zeta^{2(n-1)} \\ 1 & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{n-1} & \zeta^{2(n-1)} & \cdots & \zeta^{(n-1)^2} \end{pmatrix},$$

which is a Vandermonde matrix, with determinant

$$\prod_{0 \le i < j \le n-1} (\zeta^j - \zeta^i) \neq 0.$$

Consequently, we conclude

$$\mathbb{C}^n = V_0 \oplus V_1 \oplus \cdots \oplus V_{n-1},$$

where $V_j = \mathbb{C} v_j$ and $\rho(\sigma)$ acts on $V_j$ via multiplication by $\zeta^{-j}$.