Math 468, Homework 9

Let λ and μ denote partitions of d. Recall that the *Kostka number* $K_{\mu\lambda}$ is the number of ways you can fill the boxes of the Young diagram of μ with λ_1 1's, λ_2 2's, etc., such that the entries in each row are nondecreasing and those in each column are strictly increasing.

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we have

$$M_\lambda = \sum_{\sigma \in \mathfrak{S}_k} x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(k)}^{\lambda_k},$$

the Schur polynomial

$$S_\lambda = \det(x_j^{\lambda_i + k - i}) / \det(x_j^{k-i}),$$

and

$$H_\lambda = H_{\lambda_1} H_{\lambda_2} \cdots H_{\lambda_k}, \quad H_m(x) = \sum_{m_1 + \cdots + m_k = m} x_1^{m_1} \cdots x_k^{m_k}.$$

Finally, we have the power symmetric functions

$$P_q = x_1^q + x_2^q + \cdots + x_k^q$$

and

$$P^{(i)} = P_1^{i_1} \cdots P_d^{i_d}.$$

We have the formulas

$$S_\lambda = \det(H_{\lambda_i + j - i}) \tag{1}$$

and

$$H_\lambda = \sum_\mu K_{\mu\lambda} S_\mu. \tag{2}$$

Let W be the vector space of all symmetric polynomials of degree d in $x_1, \ldots, x_k$. Consider the bilinear pairing $\langle , \rangle$ defined so that

$$\langle H_\lambda, M_\mu \rangle = \delta_{\lambda,\mu}.$$

We proved that the S_λ are an orthonormal basis for W with respect to this form.

For further information, please consult §4.3 and A.1 of Fulton-Harris, *Representation theory*.

1)Express $S_{(1,1)}S_{(2,2)}$ as a sum of Schur polynomials.

2)Let $d \geq 6$ and consider the representation $U_\lambda = \mathbb{C}[\mathfrak{S}_d]a_\lambda$ for $\lambda = (d-3, 2, 1)$. Determine which irreducible representations of $\mathfrak{S}_d$ arise in U_λ and compute their multiplicities.

3)Show that $K_{\mu\lambda}$ is nonzero if and only if

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$$

for each i.

4)Given $i = (i_1, \ldots, i_d)$ and $j = (j_1, \ldots, j_d)$ with

$$i_1 + 2i_2 + \cdots + di_d = j_1 + 2j_2 + \cdots + dj_d = d,$$

show that

$$\left\langle P^{(i)}, P^{(j)} \right\rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1^{i_1} i_1! 2^{i_2} i_2! \ldots d^{i_d} d! & \text{if } i = j. \end{cases}$$

5)Let $a = (a_1, \ldots, a_k)$ be integers such that $a_1 + \ldots + a_k = d$. Define $\psi_a = 0$ if some $a_j < 0$. Otherwise, let ψ_a denote the character on $\mathfrak{S}_d$ induced from the trivial representation of $\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \cdots \times \mathfrak{S}_{a_k}$. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ denote a partition of d. Using equations (1) and (2), show that

$$\chi_\lambda = \sum_{\tau \in \mathfrak{S}_k} \epsilon(\tau) \psi_{\lambda_1 + \tau(1) - 1, \lambda_2 + \tau(2) - 2, \ldots, \lambda_k + \tau(k) - k}.$$