

A note on today's class

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I should have made one point more explicit in response to the question at the end of class.

Recall we were considering the mapping

$$\begin{aligned} F : \mathbb{C}[\mathfrak{S}_d] &\rightarrow \mathbb{C}[\mathfrak{S}_d] \\ x &\mapsto xc_\lambda, \end{aligned}$$

with image $V_\lambda = \mathbb{C}[\mathfrak{S}_d]c_\lambda$. We had established the following properties:

- $c_\lambda^2 = n_\lambda c_\lambda$ for some scalar n_λ ;
- $V_\lambda \simeq \text{image}(F) \neq 0$, thus

$$\dim V_\lambda + \dim \ker(F) = d!.$$

The goal was to verify that

$$n_\lambda = \frac{d!}{\dim(V_\lambda)}.$$

The following question was raised:

Why is there a direct sum decomposition $\mathbb{C}[\mathfrak{S}_d] = \ker(F) \oplus V_\lambda$?

If there were not such a decomposition, the intersection

$$\ker(F) \cap V_\lambda \neq 0.$$

This would mean that $xc_\lambda = 0$ for some $0 \neq x \in V_\lambda$, which means that $n_\lambda = 0$. In particular, we'd have

$$V_\lambda = \text{image}(F) \subset \ker(F),$$

i.e., F is *nilpotent* and $\text{tr}(F) = 0$. This contradicts our fact that $\text{tr}(F) = d!$. (We had computed this directly using the basis $\{g\}_{g \in \mathfrak{S}_d}$ for $\mathbb{C}[\mathfrak{S}_d]$.)