MATH 465 BASIC ALGEBRAIC GEOMETRY LECTURE 9

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4. Quasi-projective varieties

Notation. For any polynomial F, we shall let $F_{[j]}$ denote the homogeneous degree j part of F, $j = 0, \dots, \deg F$.

4.1 Closed subsets of projective space. Let V be a vector space of dimension $n+1$ (over k).

Definition 1. $\mathbb{P}(V)(=\mathbb{P}^n)$ is the set of lines in V through the origin O.

A point in $\mathbb{P}(V)$ is denoted by $[X_0, \dots, X_n]$, where X_i are coordinates of V.

$\mathbb{P}^n := \mathbb{P}(k^n)$ has distinguished open sets U_i (or sometimes denoted by $\mathbb{A}^n_i$) defined by

$$\{[X_0, \dots, X_n] \in \mathbb{P}^n \mid X_i \neq 0\}.$$

$U_i \simeq \mathbb{A}^n$ and $\mathbb{P}^n = \bigcup_{i=0}^n U_i$.

Let $F \in k[X_0, \dots, X_n]$, $\xi = [\xi_0, \dots, \xi_n] \in \mathbb{P}^n$.

Definition 2. F vanishes at ξ (or ξ is a zero of F) if $F(\lambda\xi_0, \dots, \lambda\xi_n) = 0$ for all $\lambda \in k^*$.

$F(\lambda\xi_0, \dots, \lambda\xi_n) = F_{[0]}(\xi_0, \dots, \xi_n) + \lambda F_{[1]}(\xi_0, \dots, \xi_n) + \cdots + \lambda^r F_{[r]}(\xi_0, \dots, \xi_n) = 0$ if and only if $F_{[i]}(\xi_0, \dots, \xi_n) = 0$ for all $i = 1, \dots, r = \deg F$.

Remark 1. $\{F = 0\} = \cap_{i=1}^{\deg F}\{F_i = 0\}$.

Definition 3. $X \subset \mathbb{P}^n$ is a *closed set* if X is the set of common zeros in $\mathbb{P}^n$ for a finite number of polynomials which we may assume homogeneous.

Definition 4. An ideal $I \subset k[X_0, \dots, X_n]$ is said to be *homogeneous* (or *graded*) if $F \in I$ if and only if $F_{[i]} \in I$ for all $i = 1, \dots, \deg F$.

Lemma 1. *An ideal is homogeneous if and only if it is generated by homogeneous polynomials.*

Proof. If $I \subset k[\underline{x}]$ is homogeneous, the homogeneous parts of the generators of I obviously generate I. Conversely, let I be an ideal generated by homogeneous polynomials f_i, $i = 1, \ldots, r$. Suppose that $g \in I$ i.e. $g = \sum_{i=1}^r a_i f_i$, $a_i \in k[\underline{x}]$. The assertion follows since each homogeneous part $(a_i)_{[j]} f_i$ of a_i is in I. ☐

Lemma 2. *Let X be a closed set in $\mathbb{P}^n$. Then $I(X) = \{F \in k[\underline{x}] \mid F(\xi) = 0, \forall \xi \in X\}$ is homogeneous.*

Recall that in affine geometry, $V(I) = \phi$ if and only if $I = k[\underline{x}]$ (Nullstellensatz). This is not true in projective case. For instance, $V((X_0, \ldots, X_n)) = \phi$ since X_i, $i = 0, \ldots, n$ do not have z common zero in $\mathbb{P}^n$.

Lemma 3. *Let $I_{\geq d} = \{$homogeneous of polynomials of degree $\geq d\}$. Then $V(I_{\geq d}) = \phi$.*

Proof. $I_{\geq d}$ contains X_i^d, $i = 0, \ldots, n$. Hence $V(I_{\geq d}) \subseteq V(X_0^d, \ldots, X_n^d) = \phi$. ☐

Lemma 4. *Let $I \subset k[X_0, \ldots, X_n]$ be a homogeneous ideal. Then $V(I) = \phi$ if and only if $I \supset I_{\geq d}$ for some d.*

Proof. Let $I = (F_1, \ldots, F_r)$ be a homogeneous ideal such that $V(I) = \phi$. Since $V(I) \cap \{X_0 \neq 0\} = \phi$, $\{F_i(1, \frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0})\}$ do not have a common zero in U_0. By Nullstellensatz, there exist $a_1, \ldots, a_r \in k[\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}]$ such that $\sum a_i F_i = 1$. Clear denominators by multiplying $X_0^{N_0}$, N_0 large enough.

$$\sum A_i(X_0, \ldots, X_n) F_i(X_0, \ldots, X_n) = X_0^{N_0} \in I.$$

Likewise, there exists N_i such that $X_i^{N_i} \in I$ for $i = 1, \ldots, n$. Therefore, I contains $X_0^N, \ldots, X_n^N$, $N = \max\{N_0, \ldots, N_n\}$.

$$\therefore I \supset I_{Nn}.$$

☐

Appendix

Tensor product of vector spaces. Let V and W be vector spaces over k. Then the *tensor product* $V \otimes_k W$ of V and W is the vector space spanned by symbols $v \otimes w$, $v \in V$, $w \in W$ with the relations

(1) $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$ for all $v \in V$ and $w_1, w_2 \in W$.
(2) $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$ for all $v_1, v_2 \in V$ and $w \in W$.

(3) $(\alpha v) \otimes w = v \otimes (\alpha w)$ for all $\alpha \in k$ and $v \in V$.

It is easy to construct $V \otimes_k W$. Let M be the vector subspace of $V \oplus W$ spanned by elements of the form:

$$(v, w_1 + w_2) - (v, w_1) - (v, w_2)$$
$$(v_1 + v_2, w) - (v_1, w) - (v_2, w)$$
$$(\alpha v, w) - (v, \alpha w),\ \alpha \in k$$

Then $V \otimes_k W$ is the quotient space $(V \oplus W)/M$.

Lemma 5. *Let X, Y be closed sets. Then $k[X \times Y] \simeq k[X] \otimes_k k[Y]$. Here $k[X]$ and $k[Y]$ are (more likely infinite dimensional) vector spaces over k.*

Exterior power of a vector space. The rth *exterior power* $\bigwedge^r V$ of a vector space V is the quotient space of $V \otimes_k V \otimes_k \cdots \otimes_k V$ (r copies) modulo the subspace generated by all elements of the form

$$v_{\sigma(1)} \otimes v_{\sigma_2} \otimes \cdots \otimes v_{\sigma(r)} - sgn(\sigma) v_1 \otimes v_2 \otimes \cdots \otimes v_r, \sigma \in S_r$$

where S_r is the symmetry group of r letters.

An element of $\bigwedge^r V$ is a finite linear combination of elements of the form

$$v_1 \wedge \cdots \wedge v_r, v_i \in V.$$

(1) $(\alpha v_1 + \beta v_1') \wedge v_2 \wedge \cdots \wedge v_r = \alpha(v_1 \wedge \cdots v_r) + \beta(v_1' \wedge \cdots \wedge v_r)$.
(2) $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(r)} = sgn(\sigma) v_1 \wedge \cdots \wedge v_r$.
(3) $v_1 \wedge \cdots \wedge v_r \neq 0$ if and only if $v_1, \dots, v_r$ are linearly independent.
(4) If $\{e_1, \dots, e_r\}$ is a basis for V then

$$\{e_{i_1 i_2 \cdots i_r} := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r} \mid i_1 < i_2 < \cdots < i_r\}$$

is a basis for $\bigwedge^r V$. In particular, $\dim \bigwedge^r V = \binom{\dim V}{r}$.

Any element in x can be written as

$$\sum_{i_1 < \cdots < i_r} p_{i_1 \cdots i_r} e_{i_1 i_2 \cdots i_r}.$$

The $p_{i_1 \cdots i_r}$ are homogeneous coordinates for the projective space $\mathbb{P}(\bigwedge^r V)$, called *Plücker coordinates.*

Definition 5. The dual vector space V^* is the vector space of linear functionals on V.

$$\begin{aligned} V^* &= Hom_k(V, k) \\ &= \{\, f : V \to k \mid f(\alpha v + \beta w) = \alpha f(v) + \beta f(w), \forall a \in k, \forall v, w \in V\} \end{aligned}$$

Let $\{e_1, \ldots, e_n\}$ be a basis for V. Then V^* has a dual basis $\{e_1^*, \ldots, e_n^*\}$ such that

$$e_i^*(e_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases} \quad .$$