Math 465 Lecture Notes

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9 Cohomology of projective varieties

9.1 Cohomology of twisting sheaves

Definition 9.1 Let $\mathcal{F}$ be a coherent $\mathcal{O}_{\mathbb{P}^n}$-module. For each integer d, the *Serre twist* $\mathcal{F}(d)$ is defined by the gluing data

$$\begin{array}{rcl} \mathcal{F}(d)|_{U_j} & := & \mathcal{F}|_{U_j} \\ \theta_{ij} : (\mathcal{F}(d)|_{U_j})|_{U_i \cap U_j} & \stackrel{\simeq}{\to} & (\mathcal{F}(d)|_{U_i})|_{U_i \cap U_j} \\ f & \mapsto & (x_j/x_i)^d f. \end{array}$$

A section $s \in \Gamma(U, \mathcal{F}(d))$ corresponds to a sequence of $s_i \in \Gamma(U \cap U_i, \mathcal{F}), i = 0, \ldots, n$, so that $s_i = (x_j/x_i)^d s_j$.

Note that $\mathcal{F}(d)$ and $\mathcal{F}$ have the same stalks. Furthermore, an $\mathcal{O}_{\mathbb{P}^n}$-linear homomorphism $\mathcal{F} \to \mathcal{G}$ gives rise to an $\mathcal{O}_{\mathbb{P}^n}$-linear homomorphism $\mathcal{F}(d) \to \mathcal{G}(d)$. Thus an exact sequence of $\mathcal{O}_{\mathbb{P}^n}$-modules

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

yields an exact sequence

$$0 \to \mathcal{F}(d) \to \mathcal{G}(d) \to \mathcal{H}(d) \to 0.$$

For example, if $Z \subset \mathbb{P}^n$ is a projective variety with ideal sheaf $\mathcal{I}_Z$ then there are exact sequences

$$0 \to \mathcal{I}_Z(d) \to \mathcal{O}_{\mathbb{P}^n}(d) \to \mathcal{O}_Z(d) \to 0$$

for each integer d.

For $d \geq 0$, recall there is a homomorphism

$$\begin{array}{rcl} \varphi : k[x_0, \ldots, x_n]_d & \to & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \\ p & \mapsto & (s_i = p/x_i^d \in \Gamma(U_i, \mathcal{O}_{\mathbb{P}^n}))_{i=0,\ldots,n} \end{array}$$

Theorem 9.2 (Cohomology of Serre twists) *1. φ is an isomorphism for each d, i.e. $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \simeq k[x_0, \ldots, x_n]_d$; in particular, there are no nontrivial sections when $d < 0$.*

2. $H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = 0$ for $0 < q < n$.

3. $H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1-d)) \simeq (k[x_0, \ldots, x_n]_d)^*$.

The remainder of this section is devoted to a proof of this result; we follow [Ha], chapter III §5.

Let $\mathcal{U} = \{U_0, \ldots, U_n\}$ denote the cover of $\mathbb{P}^n$ by distinguished open affine subsets. By our results on cohomology over affine varieties, we have

$$H^q(\mathbb{P}^n, \mathcal{F}) = H^q(\mathcal{U}, \mathcal{F})$$

for each coherent $\mathcal{O}_{\mathbb{P}^n}$-module. All our computations will be with reference to this covering, so we'll sometimes suppress the notation for the covering.

Our approach will entail computing the cohomology of non-coherent sheaf

$$\oplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(d).$$

Of course, this information is relevant because

$$H^q(\oplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(d)) = \oplus_{d \in \mathbb{Z}} H^q(\mathcal{O}_{\mathbb{P}^n}(d)).$$

We introduce some notation essential for the proof.

$$\begin{array}{rcl} S_d & = & k[x_0, \ldots, x_n]_d \\ S & = & \oplus_{d \in \mathbb{Z}} S_d \simeq k[x_0, \ldots, x_n] \\ S_{i_0 \ldots i_q} & = & S[x_{i_0}^{-1}, \ldots, x_{i_q}^{-1}] \\ (S_{i_0 \ldots i_q})_d & = & \text{degree } d \text{ part of } S_{i_0 \ldots i_q} \end{array}$$

Note that the localization $S_{i_0 \ldots i_q}$ has elements of both positive and negative degrees. One motivation for this notation is

Proposition 9.3 *We have*

$$\Gamma(U_{i_0\dots i_q}, \mathcal{O}_{\mathbb{P}^n}(d)) \simeq S_{i_0\dots i_q}(d)$$

and the restrictions

$$\Gamma(U_{i_0\dots i_q}, \mathcal{O}_{\mathbb{P}^n}(d)) \to \Gamma(U_{i_0\dots i_{q-1}}, \mathcal{O}_{\mathbb{P}^n}(d))$$

coincide with the canonical inclusions

$$(S_{i_0\dots i_q})_d \hookrightarrow (S_{i_0\dots i_{q+1}})_d.$$

proof of proposition: Recall the isomorphism of rings

$$\Gamma(U_{i_0\dots i_q}, \mathcal{O}_{\mathbb{P}^n}) \simeq k[x_0/x_{i_0}, \dots, x_{i_0-1}/x_{i_0}, x_{i_0+1}/x_{i_0}, \dots][x_{i_1}/x_{i_0}, \dots, x_{i_q}/x_{i_0}].$$

Since $\Gamma(U_{i_0\dots i_q}, \mathcal{O}_{\mathbb{P}^n}) \simeq \Gamma(U_{i_0\dots i_q}, \mathcal{O}_{\mathbb{P}^n}(d))$, we also have a compatible isomorphism of modules

$$\Gamma(U_{i_0\dots i_q}, \mathcal{O}_{\mathbb{P}^n}(d)) \simeq k[x_0/x_{i_0}, \dots, x_{i_0-1}/x_{i_0}, x_{i_0+1}/x_{i_0}, \dots][x_{i_1}/x_{i_0}, \dots, x_{i_q}/x_{i_0}].$$

However, these identifications break the symmetry among the indices $i_0, \dots i_q$; in particular, i_0 plays a special rôle. To remedy this, we express

$$\Gamma(U_{i_0\dots i_q}, \mathcal{O}_{\mathbb{P}^n}(d)) \simeq x_{i_0}^d k[x_0/x_{i_0}, \dots, x_{i_q}/x_{i_0}] = (S_{i_0\dots i_q})_d. \qquad (1)$$

The last identification follows from the fact that every $f \in (S_{i_0\dots i_q})_d$ can be written $f = x_{i_0}^d f_0$ with

$$f_0 \in (S_{i_0\dots i_q})_0 \simeq k[x_0/x_{i_0}, \dots, x_{i_0-1}/x_{i_0}, x_{i_0+1}/x_{i_0}, \dots][x_{i_1}/x_{i_0}, \dots, x_{i_q}/x_{i_0}].$$

The identification (1) is compatible with further localizations, i.e., the diagram

$$\begin{array}{ccc}
\Gamma(U_{i_0\dots i_q}, \mathcal{O}_{\mathbb{P}^n}(d)) & \hookrightarrow & \Gamma(U_{i_0\dots i_{q+1}}, \mathcal{O}_{\mathbb{P}^n}(d)) \\
\downarrow & & \downarrow \\
x_{i_0}^d k[x_0/x_{i_0}, \dots, x_{i_q}/x_{i_0}] & \hookrightarrow & x_{i_0}^d k[x_0/x_{i_0}, \dots, x_{i_q}/x_{i_0}, x_{i_{q+1}}/x_{i_0}] \\
\downarrow & & \downarrow \\
(S_{i_0\dots i_q})_d & \hookrightarrow & (S_{i_0\dots i_{q+1}})_d
\end{array}$$

is commutative, which yields the interpretation of the restrictions. □

The Čech complex computing $H^q(\mathcal{O}_{\mathbb{P}^n}(d))$ therefore takes the form

$$0 \to \prod_{i_0} (S_{i_0})_d \to \prod_{i_0<i_1} (S_{i_0 i_1})_d \to \ldots \to \prod_{i_0<\ldots<i_n} (S_{i_0\ldots,i_n})_d \to 0. \tag{2}$$

We emphasize here that the cochains are alternating. The boundary homomorphism is the alternating sum of canonical inclusions. The combined complex

$$0 \to \prod_{i_0} S_{i_0} \to \prod_{i_0<i_1} S_{i_0 i_1} \to \ldots \to \prod_{i_0<\ldots<i_n} S_{i_0\ldots,i_n} \to 0 \tag{3}$$

computes the cohomology $\oplus_{d\in\mathbb{Z}} H^q(\mathcal{O}_{\mathbb{P}^n}(d))$. Now complex (3) is a complex of S-modules with S-linear boundaries, hence $\oplus_{d\in\mathbb{Z}} H^q(\mathcal{O}_{\mathbb{P}^n}(d))$ inherits the structure of an S-module so that

$$(S)_e \cdot H^q(\mathcal{O}_{\mathbb{P}^n}(d)) \mapsto H^q(\mathcal{O}_{\mathbb{P}^n}(d+e)).$$

Remark 9.4 This discussion shows that for any coherent $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{F}$, the cohomology

$$\oplus_{d\in\mathbb{Z}} H^q(\mathcal{F}(d))$$

admits the structure of an S-module so that

$$S_e \cdot H^q(\mathcal{F}(d)) \mapsto H^q(\mathcal{F}(d+e)).$$

We compute H^0: Elements $s \in H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ correspond to sequences

$$(s_i \in (S_i)_d, i = 0, \ldots, n)$$

such that $s_i = s_j$ in S_{ij}. Since the inclusion homomorphisms are all injective, we can regard

$$H^0(\mathcal{O}_{\mathbb{P}^n}(d)) = \cap_{i=0,\ldots,n} (S_i)_d \in (S_{0\ldots n})_d = k[x_0, x_0^{-1}, \ldots, x_n, x_n^{-1}]_d.$$

Of course, elements of S_d are in this intersection and these are precisely the classes in $\mathrm{im}(\varphi)$. However, if there exists an N such that $x_i^N s \in k[x_0, \ldots, x_n]$ for each i, then the denominator of s divides each x_i^N and we must have $s \in k[x_0, \ldots, x_n]$. In particular, φ is an isomorphism.

We compute H^n: Classes $s \in H^n(\mathcal{O}_{\mathbb{P}^n}(-n-1-d))$ correspond to elements in

$$(S_{0\ldots n})_{-n-1-d} = k[x_0, x_0^{-1}, \ldots, x_n, x_n^{-1}],$$

modulo elements coming from any of the

$$(S_{0\dots\hat{j}\dots n})_{-n-1-d} = k[x_0, \dots, x_j, \widehat{x_j^{-1}}, \dots, x_n, x_n^{-1}].$$

The n-cochains admit a basis

$$x_0^{e_0} \dots x_n^{e_n}, \quad e_i \in \mathbb{Z}, e_0 + \dots + e_n = -n-1-d.$$

The n-coboundaries admit a basis

$$x_0^{e_0} \dots x_n^{e_n}, \quad e_i \in \mathbb{Z}, e_0 + \dots + e_n = -n-1-d, e_j \geq 0 \text{ for some } j.$$

Thus cohomology group $H^n(\mathcal{O}_{\mathbb{P}^n}(-n-1-d))$ admits a basis

$$x_0^{e_0} \dots x_n^{e_n}, \quad e_i \in \mathbb{Z}, e_0 + \dots + e_n = -n-1-d, e_j < 0 \text{ for every } j.$$

Of course, every such term admits a unique expression

$$x_0^{e_0} \dots x_n^{e_n} = \frac{1}{(x_0 \dots x_n) \cdot x^\mu},$$

where x^μ is an arbitrary monomial of degree d. In particular,

$$H^n(\mathcal{O}_{\mathbb{P}^n}(-n-1)) \simeq k \frac{1}{x_0 \dots x_n}$$

and

$$\dim H^n(\mathcal{O}_{\mathbb{P}^n}(-n-1-d)) = \dim k[x_0, \dots, x_n]_d.$$

The isomorphism

$$H^n(\mathcal{O}_{\mathbb{P}^n}(-n-1-d)) \simeq (k[x_0, \dots, x_n]_d)^*$$

is induced by the pairing

$$\begin{array}{rcl} H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \times H^n(\mathcal{O}_{\mathbb{P}^n}(-n-1-d)) & \to & H^n(\mathcal{O}_{\mathbb{P}^n}(-n-1-d)) \simeq k \\ (s,p) & \mapsto & s \cdot p = \dfrac{(s \cdot p)}{x_0 \dots x_n}, \end{array}$$

where $p \in S_d, s \in (S_{0\dots n})_{-n-1-d}$, and $(s \cdot p)$ is linear in s and p.

Localize the Čech complex (2) with respect to x_n. This corresponds to the cohomology of $\mathcal{O}_{\mathbb{P}^n}(d)|U_n$ computed with respect to the covering $\{U_0 \cap U_n, \dots, U_{n-1} \cap U_n, U_n\}$. For any coherent sheaf over an affine variety, the higher Čech cohomology with respect to any covering by affine open varieties is zero. Thus the higher cohomology of the localized complex is zero.

To interpret this, we need the following lemma:

Lemma 9.5 Let A be a ring, $T \subset A$ a multiplicative subset, and $M^* = (M^q, d_q)$ a complex of A modules. Let $M^*[T^{-1}] = (M^q[T^{-1}], d_q[T^{-1}])$ denote the localization of M^*. Then we have

$$H^q(M^*)[T^{-1}] = H^q(M^*[T^{-1}]).$$

proof We recall an exercise from the section on localization: For any A-linear $\psi : M \to N$, we have

$$(\ker\ \psi)[T^{-1}] \simeq \ker(\psi[T^{-1}]) \quad (\mathrm{im}\ \psi)[T^{-1}] \simeq \mathrm{im}(\psi[T^{-1}]).$$

The second isomorphism guarantees that cokernels also compute with localization

$$(\mathrm{cok}\ \psi)[T^{-1}] \simeq \mathrm{cok}(\psi[T^{-1}]).$$

Applying these to the boundary homomorphsisms

$$H^q(M^*)[T^{-1}] = \frac{\ker(d_q)}{\mathrm{im}(d_{q-1})}[T^{-1}] = \frac{\ker(d_q[T^{-1}])}{\mathrm{im}(d_{q-1}[T^{-1}])} = H^q(M^*[T^{-1}]). \quad \square$$

In particular, we obtain that

$$(\oplus_{d \in \mathbb{Z}} H^q(\mathcal{O}_{\mathbb{P}^n}(d)))\,[x_n^{-1}] = 0, \quad q > 0,$$

hence for each $\eta \in H^q(\mathcal{O}_{\mathbb{P}^n}(d))$ we have $x_n^N \eta = 0$ for $N \gg 0$.

To remainder of the proof is an induction on n. For $n = 1$ the proof of the theorem is complete, so we may assume $n > 1$. Let

$$H = \{x_n = 0\} \simeq \mathbb{P}^{n-1} \subset \mathbb{P}^n$$

be the hyperplane defined by the vanishing of the last coordinate. We have an exact sequences

$$0 \to \mathcal{I}_H \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^{n-1}} \to 0$$

and

$$0 \to \mathcal{I}_H(d) \to \mathcal{O}_{\mathbb{P}^n}(d) \to \mathcal{O}_{\mathbb{P}^{n-1}}(d) \to 0.$$

To compute $\mathcal{I}_H$, we use the following fact:

Lemma 9.6 Suppose that k is algebraically closed. Let $p \in k[x_0, \ldots, x_n]$ be irreducible and homogeneous of degree $D > 0$ and take

$$H = \{[x_0, \ldots, x_n] : p(x_0, \ldots, x_n) = 0\} \subset \mathbb{P}^n.$$

Then the ideal sheaf $\mathcal{I}_H \simeq \mathcal{O}_{\mathbb{P}^n}(-D)$.

proof of lemma Let J_0 denote the ideal of $H \cap U_0$ in $k[x_1/x_0, \ldots, x_n/x_0]$. Of course, J_0 contains $p(1, x_1/x_0, \ldots, x_n/x_0) = p/x_0^D$ and the Nullstellensatz implies that for each $f \in J_0$

$$f^\nu \in \left\langle p/x_0^D \right\rangle$$

for some ν. Since p is irreducible it follows that p/x_0^D divides f; we can conclude $J_0 = \left\langle p/x_0^D \right\rangle$.

Thus over each U_i, we have a surjective homomorphism

$$\begin{array}{rcl} \mathcal{O}_{\mathbb{P}^n}(-D)|_{U_i} & \rightarrow & \mathcal{I}_H|_{U_i} \subset \mathcal{O}_{\mathbb{P}^n}|_{U_i} \\ s_i & \mapsto & s_i \cdot p(x_0, \ldots, x_n)/x_i^D. \end{array}$$

Multiplication by a nontrivial polynomial is injective as well. These isomorphisms are compatible with the gluing relations: If we have $s_i(x_j/x_i)^D = s_j$ then

$$s_i \cdot p(x_0, \ldots, x_n)/x_i^D = s_j \cdot p(x_0, \ldots, x_n)/x_j^D.$$

Thus we obtain

$$\mathcal{O}_{\mathbb{P}^n}(-D) \stackrel{\simeq}{\rightarrow} \mathcal{I}_H. \quad \square$$

Applying the lemma with $p = x_n$, we obtain the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(d-1) \stackrel{\cdot x_n}{\rightarrow} \mathcal{O}_{\mathbb{P}^n}(d) \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(d) \rightarrow 0,$$

where the arrow designated by $\cdot x_n$ is multiplication by x_n. This induces a long exact sequence in cohomology:

$$\begin{array}{rl} 0 \rightarrow & H^0(\mathcal{O}_{\mathbb{P}^n}(d-1)) \stackrel{\cdot x_n}{\rightarrow} H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) \\ \stackrel{\partial}{\rightarrow} & H^1(\mathcal{O}_{\mathbb{P}^n}(d-1)) \stackrel{\cdot x_n}{\rightarrow} H^1(\mathcal{O}_{\mathbb{P}^n}(d)) \rightarrow H^1(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) \\ \ldots & H^{n-1}(\mathcal{O}_{\mathbb{P}^n}(d-1)) \stackrel{\cdot x_n}{\rightarrow} H^{n-1}(\mathcal{O}_{\mathbb{P}^n}(d)) \rightarrow H^{n-1}(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) \\ \stackrel{\partial}{\rightarrow} & H^n(\mathcal{O}_{\mathbb{P}^n}(d-1)) \stackrel{\cdot x_n}{\rightarrow} H^n(\mathcal{O}_{\mathbb{P}^n}(d)) \rightarrow H^n(\mathcal{O}_{\mathbb{P}^{n-1}}(d)). \end{array}$$

The inductive hypothesis allows us to extract the following information

1. $H^q(\mathcal{O}_{\mathbb{P}^n}(d-1)) \stackrel{\cdot x_n}{\rightarrow} H^q(\mathcal{O}_{\mathbb{P}^n}(d))$ is injective for $1 < q \leq n-1$ and surjective for $1 \leq q < n-1$; we are using $H^{q-1}(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) = 0$ and $H^q(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) = 0$ respectively.

2. $H^1(\mathcal{O}_{\mathbb{P}^n}(d-1)) \stackrel{\cdot x_n}{\to} H^1(\mathcal{O}_{\mathbb{P}^n}(d))$ is also injective; we are using the fact that

$$\begin{array}{ccc} r_d : H^0(\mathcal{O}_{\mathbb{P}^n}(d)) & \to & H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) \\ || & & || \\ k[x_0,\ldots,x_n]_d & \to & k[x_0,\ldots,x_{n-1}]_d \\ x_n & \mapsto & 0 \end{array}$$

is surjective.

3. $H^{n-1}(\mathcal{O}_{\mathbb{P}^n}(d-1)) \stackrel{\cdot x_n}{\to} H^{n-1}(\mathcal{O}_{\mathbb{P}^n}(d))$ is also surjective; we are using the fact that

$$\begin{array}{ccc} H^{n-1}(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) & \to & H^n(\mathcal{O}_{\mathbb{P}^n}(d-1)) \\ || & & || \\ (k[x_0,\ldots,x_{n-1}]_{-n-d})^* & \to & (k[x_0,\ldots,x_n]_{-n-d})^* \\ \lambda & \mapsto & \lambda \circ r_{-n-d} \end{array}$$

is injective.

To summarize, we find that multiplication by x_n induces an isomorphism

$$H^q(\mathcal{O}_{\mathbb{P}^n}(d-1)) \to H^q(\mathcal{O}_{\mathbb{P}^n}(d)), \quad 1 \le q \le n-1.$$

The same holds true for multiplication by powers x_n^N. However, for each $\eta \in H^q(\mathcal{O}_{\mathbb{P}^n}(d)), q > 0$, $x_n^N \eta = 0$ for $N \gg 0$. Thus

$$H^q(\mathcal{O}_{\mathbb{P}^n}(d)) = 0, \quad 1 \le q \le n-1,$$

and the proof of the theorem is complete.

9.2 Graded modules and $\mathcal{O}_{\mathbb{P}^n}$-modules

Recall that on an affine variety X, from each finitely generated $k[X]$-module M we constructed a coherent $\mathcal{O}_X$-module $\mathcal{F}(M)$. This gives an abundant supply of coherent sheaves on affine varieties. Here we will describe analogous constructions for $\mathbb{P}^n$. We retain the notation $S = k[x_0,\ldots,x_n]$ introduced above and write S_d for $k[x_0,\ldots,x_n]_d$.

Definition 9.7 A *graded* S-module is an S-module M that can be written as a sum of *graded pieces*

$$M = \oplus_{d\in\mathbb{Z}} M_d$$

compatible with the S-action, i.e., for all integers d and e we have

$$S_e \cdot M_d \subset M_{d+e}.$$

A *homomorphism* of graded S-modules $\phi : M \to N$ is an S-linear homomorphism such that

$$\phi(M_d) \subset N_d.$$

Example 9.8 Let $J = \langle F_1, \dots, F_r \rangle \subset S$ be an ideal generated by homogeneous polynomials F_j of degree d_j. Then J is a graded S-module. Consider the module of relations/syzygies

$$R(F_1, \dots, F_r) = \{(g_1, \dots, g_r) : g_1 F_1 + \dots + g_r F_R = 0\} \subset S^r.$$

This is also graded. Indeed, if $g_1 F_1 + \dots + g_r F_r = 0$ then each homogeneous piece is zero, i.e., for each d

$$(g_1)_{d-d_1} F_1 + \dots + (g_r)_{d-d_r} F_r = 0, \quad (\dagger)$$

where $(g)_\delta$ is the degree-δ homogeneous part of g. We regard $(\dagger)$ is an element of $R(F_1, \dots, F_r)_d$.

Definition 9.9 Let M be a graded S-module. For each integer d, we define the twisted module $M(d)$ by

$$M(d)_e = M_{d+e}.$$

In other words, $M(d)$ is isomorphic to M as an S module but its grading is shifted d-places.

Example 9.10

1. What are the graded homomorphisms $\phi : S \to S(d)$? Since ϕ is S-linear, ϕ is determined by its value at 1

   $$\phi(p) = f \cdot p, \quad f = \phi(1) \in k[x_0, \dots, x_n].$$

   The grading condition $\phi(S_0) \subset S(d)_0 = S_d$ implies f must be homogeneous of degree d.

2. Suppose again that $J = \langle F_1, \dots, F_r \rangle \subset S$ with F_j homogeneous of degree d_j. Then there is a surjective homomorphism of graded S-modules

   $$\begin{array}{rcl} S(-d_1) \oplus \dots \oplus S(-d_r) & \twoheadrightarrow & J \\ (g_1, \dots, g_r) & \mapsto & g_1 F_1 + \dots + g_r F_r \end{array}$$

   with kernel $R(F_1, \dots, F_r)$.

Proposition 9.11 *For each finitely generated graded S-module M, there exists a coherent $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{A}(M)$ with the following properties:*

1. $\mathcal{A}(S) = \mathcal{O}_{\mathbb{P}^n}$.

2. $\mathcal{A}(M)(d) = \mathcal{A}(M(d))$ *for each* d.

3. *For each homomorphism of graded S-modules $M \to N$, there is an $\mathcal{O}_{\mathbb{P}^n}$-linear homomorphism $\mathcal{A}(M) \to \mathcal{A}(N)$. $\mathcal{A}$ takes exact sequences of graded S-modules to exact sequences of $\mathcal{O}_{\mathbb{P}^n}$-modules.*

4. *There is a homomorphism*

$$\varphi_0(M) : M_0 \to \Gamma(\mathbb{P}^n, \mathcal{A}(M))$$

compatible with the S-module structure on

$$\oplus_{d\in\mathbb{Z}}\Gamma(\mathbb{P}^n, \mathcal{A}(M)(d)) = \oplus_{d\in\mathbb{Z}}\Gamma(\mathbb{P}^n, \mathcal{A}(M(d)))$$

from Remark 9.4. Precisely, we have

$$\varphi(M) := [\oplus_{d\in\mathbb{Z}}\varphi_d(M)] : M \to \oplus_{d\in\mathbb{Z}}\Gamma(\mathbb{P}^n, \mathcal{A}(M)(d))$$

is a homomorphism of graded S-modules.

proof: We construct $\mathcal{A}(M)$ locally on each of the distinguished open $U_i \subset \mathbb{P}^n$ and then glue these together. Consider the localization $(M[x_i^{-1}])_0$ as a module over

$$(S_i)_0 = k[x_0/x_i, \dots, x_{i-1}/x_i, x_{i+1}/x_i, \dots, x_n/x_i] \simeq \Gamma(U_i, \mathcal{O}_{U_i}).$$

We set

$$\mathcal{A}(M)_i = \mathcal{F}((M[x_i^{-1}])_0) \text{ on } U_i$$

with gluing maps

$$\theta_{ij} : \mathcal{A}(M)_j|_{U_i\cap U_j} \stackrel{\simeq}{\to} \mathcal{A}(M)_i|_{U_i\cap U_j}$$

induced by the isomorphisms

$$\Gamma(U_i\cap U_j, \mathcal{A}(M)_i) = (M[x_i^{-1}])_0[x_i/x_j] \quad \Gamma(U_i\cap U_j, \mathcal{A}(M)_j) = (M[x_j^{-1}])_0[x_j/x_i]$$

and the identification

$$(M[x_j^{-1}])_0[x_j/x_i] = (M[x_i^{-1}])_0[x_i/x_j].$$

We are using a fact from the cohomology of affine varieties: If X is affine, N is a finitely generated $k[X]$-module, and $f \in k[X]$, then

$$\Gamma(D(f), \mathcal{F}(N)) = \Gamma(X, \mathcal{F}(N))[f^{-1}].$$

We verify the properties claimed for $\mathcal{A}(M)$. First, when $M = S$ then $\mathcal{A}(S)_i = \mathcal{O}_{U_i}$,

$$\Gamma(U_i, \mathcal{A}(S)_i) = \Gamma(U_i, \mathcal{O}_{U_i}) = k[x_0/x_i, \ldots, x_n/x_i],$$

and the gluing maps are just equality of regular functions on the overlaps $U_i \cap U_j$. Second, we have isomorphisms

$$\begin{array}{ccc} (M[x_i^{-1}])_0 & \stackrel{\simeq}{\rightarrow} & (M(d)[x_i^{-1}])_0 \\ \mu_i & \mapsto & x_i^d \mu_i \end{array}$$

so the identification

$$(M(d)[x_j^{-1}])_0[x_j/x_i] = (M(d)[x_i^{-1}])_0[x_i/x_j]$$

induces an isomorphism

$$\begin{array}{ccc} (M[x_j^{-1}])_0[x_j/x_i] & \rightarrow & (M[x_i^{-1}])_0[x_i/x_j] \\ \mu_j & \mapsto & x_j^d x_i^{-d} \mu_j. \end{array}$$

However, $\mu_i = (x_j/x_i)^d \mu_j$ is the gluing rule for sections of $\mathcal{A}(M)(d)$.

For the third property, a homomorphism of graded S-modules induces graded homomorphisms on their localizations with respect to the x_i. The resulting homomorphisms of $\mathcal{O}_{U_i}$-modules

$$\mathcal{A}(M)_i \rightarrow \mathcal{A}(N)_i$$

are compatible with the gluing relations, and thus induce an $\mathcal{O}_{\mathbb{P}^n}$-linear homomorphism $\mathcal{A}(M) \rightarrow \mathcal{A}(N)$. Since localization preserves exactness, the same holds for the operator $\mathcal{A}$.

For the fourth property, the canonical map

$$M_0 \rightarrow (M[x_i^{-1}])_0$$

guarantees that each $m \in M_0$ gives compatible local sections in

$$\Gamma(U_i, \mathcal{A}(M)_i) = (M[x_i^{-1}])_0$$

and thus a global section in $\Gamma(\mathbb{P}^n, \mathcal{A}(M))$. We leave the reader to verify the compatibilty with the S-module structure. $\square$

References

[Ha] R. Hartshorne, *Algebraic geometry*, Springer Verlag, New York, 1977.