Math 465 Assignment 9: Due Wednesday, March 31

1)Let $\psi : M \to N$ be an A-linear homomorphism of A-modules and $S \subset A$ a multiplicative subset.
a)Show there is an induced $A[S^{-1}]$-linear homomorphism

$$\begin{array}{rcl} \psi[S^{-1}] : M[S^{-1}] & \to & N[S^{-1}] \\ m/s & \mapsto & \psi(m)/s \end{array}$$

so that the diagram

$$\begin{array}{ccc} M & \stackrel{\psi}{\longrightarrow} & N \\ \phi_S^M \downarrow & & \downarrow \phi_S^N \\ M[S^{-1}] & \stackrel{\psi[S^{-1}]}{\longrightarrow} & N[S^{-1}] \end{array}$$

commutes.
b)Show that localization is compatible with taking kernels and cokernels, i.e.,

$$\ker(\psi)[S^{-1}] = \ker(\psi[S^{-1}]) \quad \mathrm{cok}(\psi)[S^{-1}] = \mathrm{cok}(\psi[S^{-1}]).$$

2)Let $I \subset A$ be an ideal and $\mathfrak{m} \subset A$ a maximal ideal *not* containing I. Show that $I_{\mathfrak{m}} = A_{\mathfrak{m}}$.

3)Consider the Serre twisting sheaves $\mathcal{O}_{\mathbb{P}^n}(d_1)$ and $\mathcal{O}_{\mathbb{P}^n}(d_2)$. Recall that when $d_1, d_2 \geq 0$, homogeneous polynomials $p_j \in k[x_0, \ldots, x_n]_{d_j}, j = 1, 2$, give rise to sections $s_j \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_j))$. The product $p_1 p_2 \in k[x_0, \ldots, x_n]_{d_1+d_2}$ gives rise to a section $s_{12} \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_1 + d_2))$. Given an open $U \subset \mathbb{P}^n$ and arbitrary $\sigma_1 \in \Gamma(U, \mathcal{O}_{\mathbb{P}^n}(d_1))$ and $\sigma_2 \in \Gamma(U, \mathcal{O}_{\mathbb{P}^n}(d_2))$, define a product $\sigma_1 \cdot \sigma_2 \in \Gamma(U, \mathcal{O}_{\mathbb{P}^n}(d_1 + d_2))$ so that

$$s_{12} = s_1 \cdot s_2.$$

4)Consider an complex affine variety X with the Zariski topology and a closed subset $Z \subset X$.
a)Show that

$$\begin{array}{rcl} \mathcal{I}_Z & \subset & \mathcal{O}_X \\ \mathcal{I}_Z(U) & := & \{f \in \mathcal{O}_X(U) : f(z) = 0 \text{ for all } z \in U \cap Z\} \end{array}$$

is a $\mathcal{O}_X$-module.
b)Suppose that $x \in X$ but $x \notin Z$. Show carefully that $\mathcal{I}_{Z,x} = \mathcal{O}_{X,x}$.
For the next two parts, you may assume Z finite:
c)Show that the presheaf

$$U \mapsto \mathcal{O}_X(U)/\mathcal{I}_Z(U)$$

is automatically a sheaf on X, denoted $\mathcal{O}_Z$.
d)Compute $\Gamma(X, \mathcal{O}_Z)$.