Math 465 Assignment 1: Due Wednesday, January 21

1)The goal of this problem is to introduce some important affine varieties coming from linear algebra. Let k be an arbitrary field–there is no harm assuming $k = \mathbb{R}, \mathbb{C}$. Let $M_{m,n}(k)$ be the $m \times n$ matrices with entries in k

$$M_{m,n}(k) = \{ \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} : a_{ij} \in k \}.$$

We identify $M_{m,n}(k)$ with the affine space $\mathbb{A}^{mn}_k = k^{mn}$:

$$(a_{ij}) \mapsto (a_{11}, a_{12}, \dots, a_{1n}, a_{21}, \dots, a_{m1}, \dots, a_{mn}).$$

For $r \leq \min(m, n)$, consider the subset

$$V_r = \{A \in M_{m,n}(k) : \text{rank}(A) < r\} \subset M_{m,n}(k).$$

Prove that V_r is a closed subset of affine space, defined by the vanishing of the determinants of *all* the $r \times r$ minors of A.

Come up with your own general proof or work through the following steps:

a. The sample case $m = 2$ and $n = 3$: Prove

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

has rank ≤ 1 iff

$$a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{23} - a_{13}a_{21} = a_{12}a_{23} - a_{22}a_{13} = 0.$$

b. Let A and B be $m \times n$ matrices, and assume that B is obtained from A by elementary row operations (i.e., exchanging two rows or adding a multiple of one row to another.) If some $r \times r$ minor of A has nonzero determinant then some $r \times r$ minor of B has nonzero determinant.

c. Let B be an $m \times n$ matrix in row echelon form. Show that B has rank $< r$ iff every $r \times r$ minor has vanishing determinant.

d. Since every matrix can be put in row echelon form by applying elementary row operations, conclude the result.

e. *Optional challenge* If you know exterior algebra, use it to give a two line prove of the main result!

2)Here we consider some questions about finite sets

$$S = \{p_1, \dots, p_N\} \subset \mathbb{A}^2_{\mathbb{R}} = \mathbb{R}^2.$$

a. Let $S = \{(0,0), (1,0), (0,1), (1,1)\}$. Show that

$$I(S) = \langle x_1(x_1 - 1), x_2(x_2 - 1) \rangle \,,$$

that $\dim \mathbb{R}[S] = 4$, and that the images of $\{1, x_1, x_2, x_1x_2\}$ form a basis for $\mathbb{R}[S]$.

b. Retaining the notation of (a), show that

$$\begin{array}{rcl} \mathbb{R}[S] & \to & \mathbb{R}[S] \\ f & \mapsto & f \cdot x_1 \end{array}$$

is a linear transformation. Find its matrix with respect to the basis given above.

c. For arbitrary S, show that

$$\begin{array}{rcl} \mathrm{ev} : \mathbb{R}[S] & \to & \mathbb{R}^N \\ f & \mapsto & (f(p_1), \dots, f(p_N)) \end{array}$$

is a linear transformation and is bijective (one-to-one and onto). Conclude that $\dim \mathbb{R}[S] = N$.