Problem set 8: due April 14

Throughout, let d be a squarefree integer, and $\mathfrak{o} \subset \mathbb{Q}(\sqrt{d})$ the ring of algebraic integers.

1)Consider the ring of integers $\mathfrak{o} \subset \mathbb{Q}(\sqrt{-14})$.

a)Let P_1 and P_2 be the ramified primes over 2 and 7 respectively. Express each as an explicit product of primes relatively prime to the discriminant

$$P_i \equiv Q_1 \dots Q_k, \quad Q_j \nmid 2, 7.$$

b)Evaluate the characters $\chi_i(P_j), j = 1, 2$ explicitly using the product expression. Verify the product formula

$$\chi_1(P_j)\chi_2(P_j) = 1.$$

2)Find an explicit integer d so that $\mathrm{Cl}(\mathfrak{o})$ has 2-Sylow part

$$C_{2^{r_1}} \times C_{2^{r_2}}, \quad r_1, r_2 \geq 2.$$

3)For each of the following rings of quadratic integers, write down an explicit ambiguous ideal $\neq \mathfrak{o}$ which is strictly principal, and deduce the resulting relation among the ramified primes $P_1, \dots, P_t \subset \mathfrak{o}$:

a)$\mathfrak{o} \subset \mathbb{Q}(\sqrt{-210})$

b)$\mathfrak{o} \subset \mathbb{Q}(\sqrt{-21})$

c)$\mathfrak{o} \subset \mathbb{Q}(\sqrt{33})$

4)Let p, q be distinct primes $\equiv 1 \pmod 4$ with $(\frac{p}{q}) = -1$, and $\mathfrak{o}$ the ring of integers in $\mathbb{Q}(\sqrt{pq})$. Show that $\mathrm{Cl}_s(\mathfrak{o}) = \mathrm{Cl}(\mathfrak{o})$ and that $2 | \#\mathrm{Cl}(\mathfrak{o})$.