Potential Math 465 Projects

These are intended to be advisory in nature–feel free to choose your own topic. A reasonable subset of what is sketched here could make a perfectly good project. The quality and rigor of the exposition will weigh more heavily on the grade than length or completeness. You should strive to achieve the standard of clarity of a well-written textbook intended for your peers.

1)Read the account of invariants of finite groups in Chapter Seven of Cox, Little, and O'Shea. We work over an algebraically closed field k of characteristic zero. Let ζ_r be a primitive rth root of unity, i.e., $\zeta_r^r = 1$ but $\zeta_r^m \neq 1$ for any $m|r$. Over $\mathbb{C}$ we have

$$\zeta_r = \exp(2\pi i a/r), \quad a \in \mathbb{Z}, (a,r) = 1.$$

For some (or all!) of the following group actions on $k[x,y]$, compute the ring of invariants and the equations they satisfy.
a)The cyclic group of order r generated by

$$\sigma_r : (x,y) \to (\zeta_r x, \zeta_r^{r-1} y).$$

b)The dihedral group generated by σ_r and

$$\rho : (x,y) \to (iy, ix), \quad i^2 = -1.$$

c)The cyclic group of order four

$$(x,y) \to (ix, iy).$$

d)The cyclic group of order five

$$(x,y) \to (\zeta_5 x, \zeta_5^2 y)$$

e)Speculate on why the case where $G \subset \mathrm{SL}_2$ differs from the general case. For more information, see the article *Young Person's Guide to Canonical Singularities* by Miles Reid, in the 1985 Bowdoin conference proceeding.

2)We work over an algebraically closed field k of characteristic zero. We consider differential operators with constant coefficients

$$D := \sum_{i_1,\ldots,i_n} a_{i_1\ldots i_n} \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}}.$$

Do some or all of the following.
a)Show that the contant coefficient differential operators

$$R = k[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$$

form a commutative ring, isomorphic to a polynomial ring in the variables $\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n = \frac{\partial}{\partial x_n}$.
b)Let $W \subset k[x_1, \ldots, x_n]$ be a vector space of polynomials. Let

$$\text{ann}(W) = \{D \in R : Df = 0 \text{ for each } f \in W\}.$$

Show that $\text{ann}(W)$ is an ideal in R.
c)Assume that W is finite dimensional. Show that the variety $V(\text{ann}(W))$ is the origin. *Hint:* If all the polynomials in W have degree $< d$ then differential operators of degree $\geq d$ are automatically in $\text{ann}(W)$.
d)Let $I \subset R$ be an ideal and assume that $V(I)$ is the origin. Show that I contains $\mathfrak{m}^d$ for some d, where

$$\mathfrak{m} = \langle \partial_1, \ldots, \partial_n \rangle ,$$

and R/I is finite dimensional. Show that

$$\Sigma(I) = \{f \in k[x_1, \ldots, x_n] : Df = 0 \text{ for each } D \in I\}$$

is a finite dimensional vector space.
e)Under the assumptions of (d), show that R/I has the same dimension as $\Sigma(I)$.
f)Set $n = 2$ and consider the polynomial $f = x_1^2 + x_2^2$. Compute $\text{ann}(f)$ and $\Sigma(\text{ann}(f))$. Do the same for $f = x_1^3 - x_2^2$.
g)Retain the assumptions of (d). Show that $\Sigma(I)$ is *translation invariant,* i.e., if $f \in \Sigma(I)$ then $f(x_1 + a_1, x_2 + a_2, \ldots, x_n + a_n) \in \Sigma(I)$ for each $(a_1, \ldots, a_n)$. *Hint:* Use the chain rule.
h)Let $f \in k[x_1, \ldots, x_n]$. Show there is a unique, minimal translation invariant subpsace containing f. *Hint:* Expand $f(x_1 + a_1, \ldots, x_n + a_n)$ out as a polynomial in $k[a_1, \ldots, a_n][x_1, \ldots, x_n]$.
i)Let $W \subset k[x_1, \ldots, x_n]$ be a finite dimensional translation-invariant subspace. Assume that W contain a *cyclic vector*, i.e., a polynomial f such that W is the minimal translation invariant subspace containing f. Show that

$W = \Sigma(I)$ for some ideal I containing $\mathfrak{m}^d$ for some d. *Hint:* Consider the differential operators annihilating f.
For more information, see my joint paper with Y. Tschinkel, *Geometry of equivariant compactification of* $\mathbb{G}_a^n$.

3)Compute equations for the images of the following maps. Use Gröbner bases for the first few cases. For the general case, write the equations in determinantal/matrix form.
a)The rational normal curve of degree d:

$$\begin{aligned} \phi \ : \ & k^2 \to k^{d+1} \\ & (s,t) \to (s^d, s^{d-1}t, s^{d-2}t^2, \dots, t^d). \end{aligned}$$

b)The two-fold Veronese imbedding:

$$\begin{aligned} \phi \ : \ & k^{n+1} \to k^{\binom{n+2}{2}} \\ & (s_0, s_1, \dots, s_n) \to (s_0^2, s_0, s_1, \dots, s_{n-1}s_n, s_n^2). \end{aligned}$$

Hint: Use symmetric matrices.
c)The Segre threefold

$$\begin{aligned} \phi \ : \ & k^5 \to k^6 \\ & (s,t,x,y,z) \to (sx, sy, sz, tx, ty, tz). \end{aligned}$$

4)Let $f_1, \dots, f_{r_1}$ be *homogeneous* polynomials in $R = k[x_0, \dots, x_n]$ with $I := \Sigma_0(I) = \langle f_1, \dots, f_{r_1} \rangle$. Recall we showed there is a surjection of R-modules

$$\begin{aligned} q \ : \ & R^{r_1} \to \Sigma_0(I) \\ & (g_1, \dots, g_{r_1}) \to f_1 g_1 + \dots + f_{r_1} g_{r_1} \end{aligned}$$

with kernel $\Sigma_1(I)$ equal to the *module of syzygies* for I.
a)If you haven't already done so, show that *any* submodule of R^{r_1} is finitely generated. Look at the hints in prior problem sets!
b)Assume that $\Sigma_1(I)$ has r_2 homogeneous generators, i.e., with the g_i homogeneous. Show there is a surjection of R-modules

$$R^{r_2} \to \Sigma_1(I).$$

The kernel $\Sigma_2(I)$ is called the module of higher order syzygies and is finitely generated.
c)Iterating in this way we obtain a sequence

$$0 \to \Sigma_j(I) \to R^{r_j} \to R^{r_{j-1}} \to \dots R^{r_2} \to R^{r_1} \to R \to R/I \to 0.$$

Prove the *Hilbert syzygy theorem*, i.e., that $\Sigma_j(I)$ is free for some $j < n$ so the procedure above terminates after a finite number of steps. The proof is sketched in Eisenbud's *Commutative Algebra* using Gröbner basis techniques. The proof involves developing the notion of a Gröbner basis for an arbitrary submodule of R^r.
d)Compute the full sequence of syzygies for

$$f_1 = x_1^2 - x_0x_2, \quad f_2 = x_2^2 - x_0x_3, \quad f_3 = x_0x_3 - x_1x_2$$

and

$$f_1 = x_0x_4 - x_1x_3, \quad f_2 = x_0x_5 - x_2x_3, \quad f_3 = x_1x_5 - x_2x_4.$$