Math 465 Assignment 7: Due Monday, October 31

1)Let F be the quotient field of

$$\mathbb{C}[x, y, z]/\left\langle x^2 + y^2 + z^2 - 1\right\rangle.$$

Exhibit a transcendence base $w_1, \ldots, w_d$ for F over $\mathbb{C}$, and express F explicitly as an algebraic extension over $\mathbb{C}(w_1, \ldots, w_d)$.

2)Let F/k be a finitely generated field extension, i.e., $F = k(\alpha_1, \ldots, \alpha_n)$ for some $\alpha_1, \ldots, \alpha_n \in F$. Show that any two transcendence bases for F have the same number of elements.

3)Let R and S be Noetherian integral domains with $R \subset S$. Suppose that $\alpha, \beta \in S$ are roots of the monic polynomials

$$x^2 + a_1 x + a_0, x^2 + b_1 x + b_0 \in R[x]$$

respectively. Using Gröbner basis techniques, exhibit a monic polynomial that has $\alpha + \beta$ as a root. Do the same for $\alpha\beta$.

4)Show that every maximal ideal $\mathfrak{m} \subset \mathbb{R}[x_1, x_2]$ is one of the following:

- $\mathfrak{m} = \langle x_1 - \alpha_1, x_2 - \alpha_2\rangle$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$;
- $\mathfrak{m} = \langle x_2 - rx_1 - s, x_1^2 + bx_1 + c\rangle$ for some $r, s, b, c \in \mathbb{R}$ with $b^2 - 4c < 0$;
- $\mathfrak{m} = \langle x_1 - t, x_2^2 + bx_2 + c\rangle$ for some $t, b, c \in \mathbb{R}$ with $b^2 - 4c < 0$.

5)Let $I = \langle f_1, f_2\rangle \subset \mathbb{C}[x, y]$ be an ideal generated by a linear and an irreducible quadratic polynomial. Suppose that $g \in \sqrt{I}$. Show that $g^2 \in I$.