due **April 23, 2008**

Do problems 2, 4, 15, 16, 20, 21 from §17.2

The purpose of this exercise is to construct a counterexample to Exercise 17.1.32 from Dummit and Foote. Consider the ring

$$R = \mathbb{Z}[u, v, x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots]/I$$

where

$$I = \langle x_i v - y_i u, i \in \mathbb{N} \rangle .$$

Let M be the R-module

$$M = R / \langle u, v \rangle .$$

1. Consider the complex

$$R^{\oplus \mathbb{N}} \xrightarrow{C} R^{\oplus \mathbb{N}} \xrightarrow{B} R^2 \xrightarrow{A} R \to M \to 0$$

with

$$\begin{aligned} A &= \begin{pmatrix} u & -v \end{pmatrix} \\ B &= \begin{pmatrix} v & y_1 & y_2 & y_3 & \ldots \\ u & x_1 & x_2 & x_3 & \ldots \end{pmatrix} \\ C &= \begin{pmatrix} x_1 & y_1 & x_2 & y_2 & \ldots \\ -u & -v & 0 & 0 & \ldots \\ 0 & 0 & -u & -v & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} . \end{aligned}$$

Show that this is exact.

2. Let

$$N = R / \langle u, v, x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots \rangle \simeq \mathbb{Z}$$

and apply $\mathrm{Hom}_R(-, N)$ to the resolution above to compute the groups $\mathrm{Ext}^i_R(M, N)$ for $i = 0, 1, 2$. Show that $\mathrm{Ext}^2_R(M, N) \simeq \mathbb{Z}^{\times \mathbb{N}}$.

3. Let p be a prime integer and $D = \langle p^e : e \geq 0 \rangle$. Show that the canonical homomorphism

$$D^{-1}\mathrm{Ext}^2_R(M, N) \to \mathrm{Ext}^2_{D^{-1}R}(D^{-1}M, D^{-1}N)$$

fails to be surjective.

Conclude that $\mathrm{Ext}(M, -)$ fails to commute with localization even when M is finitely presented.