Math 428 Lecture Notes

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The remainder of the course will cover in depth one topic traditionally included in a second course in complex analysis: Elliptic functions. Elliptic functions were one of the crowning glories of nineteenth century mathematics, and have found many applications in physics and engineering. They were the well-spring for many important topics in twentieth century mathematics, including elliptic curves and modular forms. Recently, they have returned as objects of interest in their own right. From the enumerative geometry of rational curves to motivic cohomology, elliptic functions are motivating examples and important computational tools.

By way of introduction, let's explore how some basic calculus problems can lead to interesting new functions. These examples are borrowed from chapter six of

> Peter L. Walker, *Elliptic functions: A constructive approach*, John Wiley & Sons, 1996.

Arclength of ellipses

Consider an ellipse with major and minor arcs $2a$ and $2b$ and eccentricity $e := (a^2 - b^2)/a^2 \in [0, 1)$, e.g.,

$$(x/a)^2 + (y/b)^2 = 1.$$

What is the arclength $\ell(a, b)$ of the ellipse, as a function of a and b?

There are two easy observations to be made:

1. $\ell(ra, rb) = r\ell(a, b)$, because rescaling by a factor r increases the arclength by the same factor;

2. $\ell(a,a) = 2\pi a$, because we know the circumference of a circle.

Of course, π is transcendental so it is debatable how well we understand it!

The total arclength is four times the length of the piece in the first quadrant, where we have the relations

$$y = b\sqrt{1-(x/a)^2}, \quad dy/dx = \frac{-(xb/a^2)}{\sqrt{1-(x/a)^2}}.$$

Thus we obtain

$$\begin{aligned}
\ell(a,b) &= 4\int_0^a \sqrt{1+(dy/dx)^2}\,dx \\
&= 4\int_0^a \sqrt{1+\frac{(x/a)^2(b/a)^2}{1-(x/a)^2}}\,dx \\
&= 4\int_0^a \sqrt{\frac{1-e(x/a)^2}{1-(x/a)^2}}\,dx \\
&\quad \text{substituting } z = (x/a) \\
&= 4a\int_0^1 \sqrt{\frac{1-ez^2}{(1-z^2)}}\,dz \\
&= 4a\int_0^1 \frac{1-ez^2}{\sqrt{(1-z^2)(1-ez^2)}}\,dz.
\end{aligned}$$

This is an example of an *elliptic integral of the second kind.*

The simple pendulum

How do we compute the period of motion of a simple pendulum?

Suppose the length of the pendulum is L and the gravitational constant is g. Let θ be the angle of the displacement of the pendulum from the vertical. The motion of the pendulum is governed by a differential equation

$$\theta'' = -g/L \sin\theta.$$

In college calculus and physics classes, this is traditionally linearized to

$$\theta'' = -(g/L)\theta,$$

so that the solutions take the form

$$\theta(t) = A\cos\omega t + B\sin\omega t, \quad \omega = \sqrt{g/L}.$$

We obtain simple harmonic motion with frequency ω and period $2\pi/\omega$.

We shall consider the nonlinear equation, using a series of substitutions. First, note that our equation integrates to

$$(\theta')^2/2 - \omega^2\cos\theta = \text{constant}.$$

Assume that $\theta(0) = 0$ and $\theta'(0) > 0$. Let $t_0 > 0$ be the first local maximum of θ, i.e., $\theta'(t_0) = 0$ and $\alpha = \theta(t_0)$ is the maximal displacement of the pendulum. Thus we have

$$(\theta')^2/2 - \omega^2\cos\theta = -\omega^2\cos\alpha$$

and thus

$$\theta' = \pm\omega\sqrt{2(\cos\theta - \cos\alpha)}.$$

We must take the positive square root for $t \in (0, t_0)$. Integrating again, we obtain

$$\omega t = \int_0^\theta \frac{d\varphi}{\sqrt{2(\cos\varphi - \cos\alpha)}} = 1/2\int_0^\theta \frac{d\varphi}{\sqrt{\sin^2(\alpha/2) - \sin^2(\varphi/2)}}.$$

Substituting

$$z = \frac{\sin(\varphi/2)}{\sin(\alpha/2)}, \quad \rho = \sin(\theta/2)/\sin(\alpha/2), \quad e = \sin^2(\alpha/2) \in [0,1)$$

we obtain

$$\omega t = \int_0^\rho \frac{dz}{\sqrt{(1-z^2)(1-ez^2)}}.$$

At maximal displacement $\theta = \alpha$ we have $\rho = 1$, so the first time where maximal displacement occurs is given by

$$1/\omega\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-ez^2)}}.$$

The *period* of the oscillation is four times the time needed to achieve the maximal displacement

$$4/\omega\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-ez^2)}}.$$

These are examples of *elliptic integrals of the first kind.*

Finally, we should point out that actually computing the function $\theta(t)$ involves inverting the function

$$\rho \to \int_0^\rho \frac{dz}{\sqrt{(1-z^2)(1-ez^2)}}.$$

Connection to complex analysis

The common denominator of the examples above is the expression

$$f(z) := \sqrt{(1-z^2)(1-ez^2)}, \quad 0 \leq e < 1.$$

As a real-valued function, this makes sense in the range

$$(-\infty, -1/\sqrt{e}] \cup [-1, 1] \cup [1/\sqrt{e}, \infty),$$

where the argument of the square root is positive. Of course, we should also specify whether we are taking the positive square root f_+ or the negative square root f_-.

After a course in complex analysis, we naturally see $f(z)$ as a *multiply-valued* function of the cut complex plane, where the cuts are the intervals

$$(-1/\sqrt{e}, -1), (1, 1/\sqrt{e}).$$

In other words, in the open subset

$$\mathbb{C} - [(-1/\sqrt{e}, -1) \cup (1, 1/\sqrt{e})]$$

we can define two functions f_+ and f_-, so that $f_+ = -f_-$ and

$$f_+^2 = f_-^2 = (1-z^2)(1-ez^2).$$

Alternately, we can think of this as a single-valued function on the *Riemann surface* S obtained by gluing two copies of the cut complex plane together along the cuts. The function f is defined on S by setting it equal to f_+ on the first cut plane and f_- on the second cut plane.

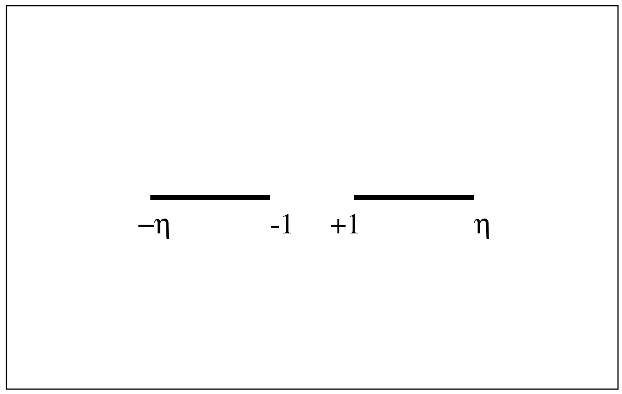

Figure 1: Cut complex plane, with $\eta = 1/\sqrt{e}$

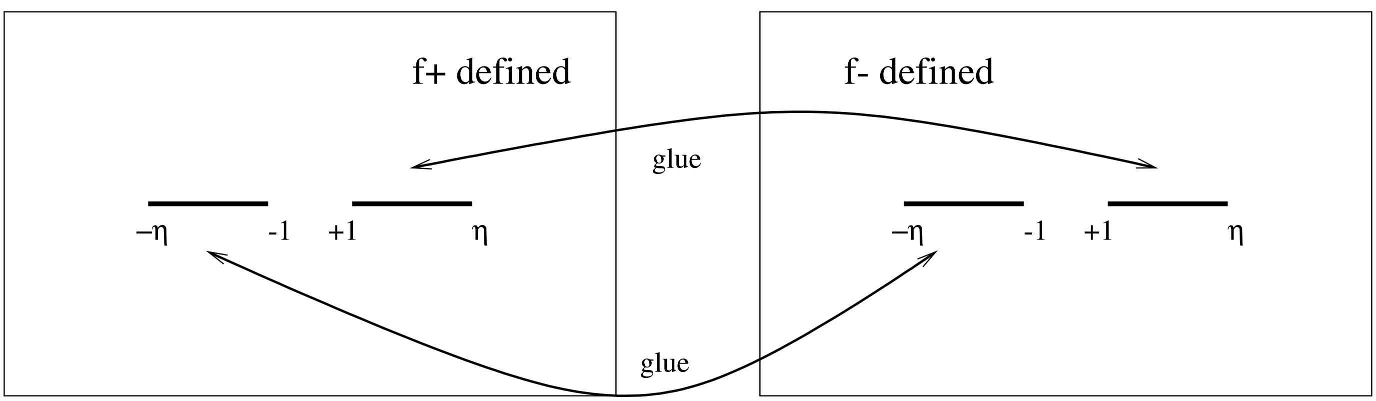

Figure 2: Associated Riemann surface