Math 428, Assignment 12: due December 5

1)Fix $\ell \in \mathbb{N}$ and let a, b be rational numbers so that $\ell a, \ell b \in \mathbb{Z}$. Consider *theta functions with rational characteristics*

$$\vartheta_{a,b}(v,\tau) = \sum_{n=-\infty}^{n=\infty} \exp[\pi i((a+n)^2\tau + 2(n+a)(v+b))].$$

a)Show that $\vartheta_{0,0} = \theta_3$, $\vartheta_{0,1/2} = \theta_2$, $\vartheta_{1/2,0} = \theta_1$, and $\vartheta_{1/2,1/2} = -\theta_0$.

b)Show that

$$\vartheta_{a,b}(v,\tau) = \exp[\pi i(a^2\tau + 2a(v+b))]\vartheta_{0,0}(v + a\tau + b, \tau).$$

c)Show that $\vartheta_{a,b} = \vartheta_{a+N,b}$ and

$$\vartheta_{a,b+N}(v,\tau) = \vartheta_{a,b}(v+N,\tau) = \exp[2\pi i a N]\vartheta_{a,b}(v,\tau).$$

Conclude that for fixed ℓ, the theta functions with rational characteristics span a vector space of complex dimension $\leq \ell^2$.

d)Describe the set of all zeros of $\vartheta_{a,b}(v,\tau)$ (as a function of v for fixed τ).

2)Recall that complex projective space $\mathbb{C}\mathrm{P}^3$ is defined as the set of equivalence classes $[z_0, z_1, z_2, z_3]$, $z_j \in \mathbb{C}$, where

$$[z_0, z_1, z_2, z_3] \sim [w_0, w_1, w_2, w_3] \text{ if for some } \lambda \neq 0, w_j = \lambda z_j, j = 0, 1, 2, 3.$$

Consider the complex torus $E = \mathbb{C}/\Lambda$, where Λ is a lattice with basis $\omega_1 = 2, \omega_2 = 2\tau$. Show that

$$\Theta_\tau(v) := [\theta(v,\tau), \theta_1(v,\tau), \theta_2(v,\tau), \theta_3(v,\tau)]$$

yields a well-defined map $\Theta_\tau : E \to \mathbb{C}\mathrm{P}^3$. Does it yield a well-defined map $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) \to \mathbb{C}\mathrm{P}^3$?

3)Verify the identities

$$\begin{aligned} \theta_3(v,\tau)^2\theta_3(0,\tau)^2 &= \theta_2(v,\tau)^2\theta_2(0,\tau)^2 + \theta_1(v,\tau)^2\theta_1(0,\tau)^2 \\ \theta(v,\tau)^2\theta_3(0,\tau)^2 &= \theta_2(v,\tau)^2\theta_1(0,\tau)^2 - \theta_1(v,\tau)^2\theta_2(0,\tau)^2. \end{aligned}$$

Conclude that the image $\Theta_\tau(E)$ satisfies the quadratic equations

$$z_3^2\theta_3(0,\tau)^2 = z_2^2\theta_2(0,\tau)^2 + z_1^2\theta_1(0,\tau)^2 \quad z_0^2\theta_3(0,\tau)^2 = z_2^2\theta_1(0,\tau)^2 - z_1^2\theta_2(0,\tau)^2. \tag{1}$$

4)Show that
$$\theta_3(0,\tau)^4 = \theta_1(0,\tau)^4 + \theta_2(0,\tau)^4.$$
Square the equations 1 and add them together to conclude
$$z_0^4 + z_3^4 = z_2^4 + z_1^4.$$
In particular, we have the identity
$$\theta(v,\tau)^4 + \theta_3(v,\tau)^4 = \theta_2(v,\tau)^4 + \theta_1(v,\tau)^4.$$
Remark: In language of modular curves, we are finding equations for the universal curve over the elliptic curves with order four torsion.