

Math 428, Assignment 11: due November 26

1) Consider the period lattice Λ generated by the basic periods

$$\omega'_1 = 7 + 8i \quad \omega'_2 = 3 + 4i$$

and set

$$\tau' = \omega'_2/\omega'_1.$$

a) Find an element γ in the modular group Γ so that $\gamma(\tau')$ is in the domain

$$D = \{\tau \in \mathcal{H} : |\tau| \geq 1, -1/2 \leq \operatorname{Re}(\tau) \leq 1/2\}.$$

b) Express γ as a product of the generators

$$S : \tau \rightarrow -1/\tau \quad T : \tau \rightarrow \tau + 1.$$

In Chandrasekharan, these are denoted B and A respectively.

c) Find primitive periods ω_1 and ω_2 of the lattice Λ .

2) Let $\Lambda \subset \mathbb{C}$ be a lattice generated by primitive periods ω_1, ω_2 . We say that Λ admits *complex multiplication* if there is a complex number α , $\alpha \notin \mathbb{Z}$, so that

$$\alpha\Lambda := \{\alpha\omega : \omega \in \Lambda\} \subset \Lambda.$$

a) Show that $\alpha \notin \mathbb{R}$.

b) Show the multiplication map

$$\begin{aligned} \mu_\alpha : \Lambda &\rightarrow \Lambda \\ \omega &\rightarrow \omega\alpha \end{aligned}$$

can be expressed, in terms of the basis ω_1, ω_2 , as a 2×2 matrix with integer entries.

c) Find an example of a lattice with complex multiplication by $\alpha = \sqrt{-2}$; write down a matrix for $\mu_{\sqrt{-2}}$.

d) Assume there exists a lattice with complex multiplication by α . Show that α is an *imaginary quadratic integer*, i.e., it satisfies an equation

$$\alpha^2 + a\alpha + b = 0, \quad a, b \in \mathbb{Z}.$$

Hint: What is the characteristic polynomial for the matrix of μ_α ?

e) *Extra credit:* If α is an imaginary quadratic integer, show there exists a lattice with complex multiplication by α .

3) Fix a positive integer N and consider

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}.$$

a) Show that $\Gamma_0(N)$ is a subgroup of Γ , called a *congruence subgroup*.

b) Show that the index of $\Gamma_0(2)$ in Γ is equal to three, with cosets

$$\Gamma_0(2), S\Gamma_0(2), T\Gamma_0(2), \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

c) Describe a fundamental domain for $\Gamma_0(2)$.

4) For a lattice Λ , recall that

$$g_2(\Lambda) = 60 \sum_{\omega \neq 0 \in \Lambda} \omega^{-4} \quad g_3(\Lambda) = 140 \sum_{\omega \neq 0 \in \Lambda} \omega^{-6}.$$

Let \wp_Λ denote its Weierstrass function.

a) For $\Lambda = \langle 1, i \rangle$, show that $g_3(\Lambda) = 0$. *Hint:* Consider the complex multiplication on Λ

$$\omega \rightarrow i\omega.$$

b) For $\Lambda = \langle 1, \zeta_3 \rangle$, where $\zeta_3 = e^{2\pi i/3}$, show that $g_2(\Lambda) = 0$.

c) Show that there exists no lattice Λ with

$$g_2(\Lambda) = g_3(\Lambda) = 0.$$

5) Show that each derivative $\wp_\Lambda^{(n)}$, $n > 1$, can be expressed as a polynomial in $\mathbb{Q}[\wp_\Lambda, \wp'_\Lambda, g_2(\Lambda)]$.

6) Consider a lattice Λ with primitive periods ω_1 and ω_2 and period parallelogram P . For each $n \in \mathbb{N}$ consider

$$\phi(z) := \wp_\Lambda(nz), \quad n \in \mathbb{N}.$$

Show that ϕ is elliptic, compute its periods, and describe a period parallelogram.

7) Consider the lattice $\Lambda = \langle 1, \sqrt{-2} \rangle$. Show that the function

$$\psi(z) := \wp_{\Lambda}(\sqrt{-2}z)$$

is elliptic for Λ . Compute the order of ψ , as an elliptic function for Λ , and identify its poles in the region

$$P = \{x + y\sqrt{-2} : 0 \leq x < 1, 0 \leq y < 1\}.$$