Math 428, Assignment 10
1)Consider the integral expression (cf. notes to chapter 1 of Chandrasekharan)

$$I(u) := \int_0^u \frac{dz}{\sqrt{(1-z^2)(1+z^2)}}.$$

Prove the *doubling formula* of Fagnano

$$I(r) = 2I(u), \quad r = \frac{2u\sqrt{1-u^4}}{1+u^4}.$$

Hint: What happens to

$$\frac{dz}{\sqrt{(1-z^2)(1+z^2)}}$$

under the substitution

$$z = \frac{2w\sqrt{1-w^4}}{1+w^4}?$$

Remark: The complex plane curve

$$C = \{(y,z) : y^2 = 1 - z^4\} \subset \mathbb{C}^2$$

is an *elliptic curve* and admits a group law. Multiplication by two in the group is given by the formula

$$(y,z) \to \left(\frac{1-6z^4+z^8}{(1+z^4)^2}, \frac{2yz}{1+z^4}\right).$$

2)Verify the formula

$$\cot z = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right).$$

Here is one possible approach: Consider the integral

$$I = \frac{1}{2\pi i}\int_C (\cot\zeta)\left(\frac{1}{z-\zeta} + \frac{1}{\zeta}\right)d\zeta$$

where the contour C is defined in Figure 1. Assume that z is contained in the interior of C and $z \neq n\pi$ for any $n \in \mathbb{Z}$.

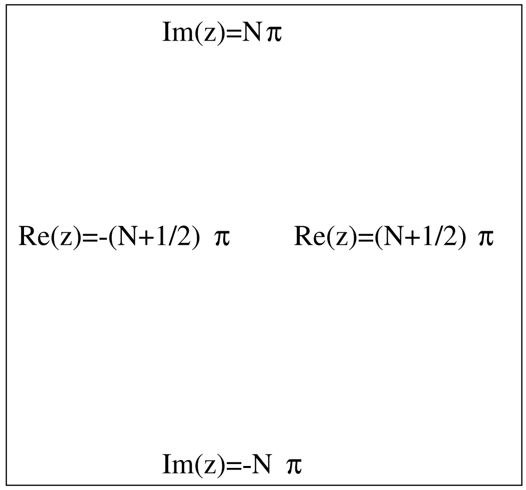

Figure 1: The contour C

a)Show by the residue theorem that

$$I = -\cot z + \frac{1}{z} + \sum_{n\neq 0, -N\leq n\leq N} (\frac{1}{z-n\pi} + \frac{1}{n\pi}).$$

b)Show that $|\cot \zeta| \leq 1$ on the vertical parts of the contour and $|\cot \zeta| \leq \coth \pi$ on the horizontal parts of the contour.
c)Establish the bound

$$|I| \leq c(4N+1)(\coth \pi)\frac{|z|}{N\pi(N\pi - |z|)}$$

for some constant c. Conclude that $\lim_{N\to\infty} I = 0$.

3)Show that

$$\frac{\pi^2}{6} = \sum_{n\geq 1}\frac{1}{n^2}.$$

Hint: Evaluate the derivative at zero for both sides of the expression

$$\cot z - \frac{1}{z} = \sum_{n\neq 0}(\frac{1}{z-n\pi} + \frac{1}{n\pi}).$$

4)Give a careful proof that

$$F(z) = \sum_{(m,n)\in\mathbb{Z}^2} (z-(m+ni))^{-k}$$

defines an elliptic function for each integer $k \geq 3$.