Glossary of topological terms used in the definition of Riemann surfaces

A *topological space* is a set X along with a collection of subsets

$$\{U \subset X\},$$

called the *open subsets of* X, satisfying the following conditions:

- $\emptyset$ and X are open;
- if U_1 and U_2 are open then $U_1 \cap U_2$ is open;
- if $\{U_i\}_{i \in I}$ is an arbitrary collection of open subsets then $\cup_{i \in I} U_i$ is open.

A *closed subset* $Z \subset X$ is a set with open complement $X \setminus Z$.

For example, let $X = \mathbb{R}^n$ and define $U \subset X$ to be open if, for each $x \in U$, there exists an $\epsilon > 0$ so that

$$D(x, \epsilon) := \{y : \|y - x\| < \epsilon\} \subset U.$$

In other words, each point of U admits a small neighborhood entirely contained in U.

A map of topological spaces $\phi : X \to Y$ is *continuous* if, for each open $V \subset Y$, the preimage

$$\phi^{-1}(V) := \{x \in X : \phi(x) \in V\}$$

is open.

You should check that a map $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if, for each $x_0 \in \mathbb{R}^n$ and $\epsilon > 0$, there exists a $\delta > 0$ so that

$$D(x_0, \delta) \subset \phi^{-1} D(\phi(x_0), \epsilon).$$

A *homeomorphism* $\phi : X \overset{\sim}{\to} Y$ is a continuous map of topological spaces with continuous inverse.

A topological space X is *Hausdorff* if, for any $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exist open subsets $U_1, U_2 \subset X$ with $U_1 \ni x_1, U_2 \ni x_2$, and $U_1 \cap U_2 = \emptyset$.

For instance, $\mathbb{R}^n$ is Hausdorff with the topology introduced above. Given $x_1, x_2 \in \mathbb{R}^n$, let $\epsilon = \|x_1 - x_2\|/2$ and take $U_1 = D(x_1, \epsilon)$ and $U_2 = D(x_2, \epsilon)$.

A *basis* of a topological space X is a collection $\{U_b\}_{b \in \mathcal{B}}$ of open subsets of X with the following property: For each $x \in X$ and open set $V \ni x$, there exists a basis element U_b with

$$x \in U_b \subset V.$$

For example, the open balls in $\mathbb{R}^n$

$$\{D(x, \epsilon), x \in \mathbb{R}^n, \epsilon \in \mathbb{R}_>\}$$

form a basis, essentially by definition. However, a more economical basis is

$$\{D(X, \epsilon), x \in \mathbb{Q}^n, \epsilon \in \mathbb{Q}_>\};$$

this has a countable number of elements.

A topological space is *second countable* if it admits a countable basis.

A topological space X is *connected* if the only sets that are both open and closed are $\emptyset$ and X. It is *pathwise connected* if, for any $x_1, x_2 \in X$, there exists a continuous map from an interval

$$\gamma : [0, 1] \to X, \quad \gamma(0) = x_1, \gamma(1) = x_2.$$

It is a theorem that every pathwise-connected topological space is connected.

A topological space X is *compact* if, for any collection of open subsets $\{U_i\}_{i \in I}$ with $X \subset \cup_{i \in I} U_i$, there exist a finite set of indices $i_1, \ldots, i_N \in I$ so that $X \subset \cup_{j=1,\ldots,N} U_{i_j}$. In other words, every covering of X admits a finite subcovering.

If $\phi : X \to Y$ is a continuous map of topological spaces and X is compact then $\phi(X) \subset Y$ is also compact. A standard result from advanced calculus, the *Heine Borel theorem*, says that every closed bounded subset of $\mathbb{R}^n$ is compact.