

Math 428 Assignment 5, due October 9

Do problems III.3.H, III.3.I, III.3.J from Miranda and the following:

1) Consider the action

$$\begin{aligned}\mathbb{C}^+ \times \mathbb{C}\mathbb{P}^1 &\rightarrow \mathbb{C}\mathbb{P}^1 \\ a \cdot [x_0 : x_1] &= \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} [x_0 : x_1] = [x_0 + ax_1 : x_1].\end{aligned}$$

- Show this is an effective holomorphic action of the additive group $G := \mathbb{C}^+$ on $\mathbb{C}\mathbb{P}^1$.
- Describe the topology on the quotient $\mathbb{C}\mathbb{P}^1/G$.
- Compute G_p for $p = [1 : 0]$. Describe the homomorphism

$$a_1 : G_p \rightarrow \mathbb{C}^\times$$

constructed in Proposition 3.1 on page 76 of Miranda.

2) Consider the group

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}, \det(A) = 1 \right\}$$

and the subgroup

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{Z}, \det(A) = 1 \right\}.$$

Regard the upper half-plane

$$H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} \subset \mathbb{C} \subset \mathbb{C}\mathbb{P}^1$$

as an open subset of $\mathbb{C}\mathbb{P}^1$.

- Show that the matrix action

$$\begin{aligned}\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}\mathbb{P}^1 &\rightarrow \mathbb{C}\mathbb{P}^1 \\ A \cdot [x_0 : x_1] &= [a_{11}x_0 + a_{12}x_1 : a_{21}x_0 + a_{22}x_1]\end{aligned}$$

induces a holomorphic action

$$\begin{aligned}\mathrm{SL}_2(\mathbb{R}) \times H &\rightarrow H \\ A \cdot z &= \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}}.\end{aligned}$$

- b. Show that the $\mathrm{SL}_2(\mathbb{R})$ -action is transitive, i.e., for each $z \in H$ we have $\mathrm{SL}_2(\mathbb{R})z = H$.
- c. Compute the stabilizer of $i \in H$.
- d. Consider the induced $\mathrm{SL}_2(\mathbb{Z})$ -action on H . Show that each point has stabilizer equal to a finite cyclic group.
- e. Prove that the cyclic groups that arise have orders $m = 2, 4$, or 6 . Give examples in each case.