

a. Show that the matrix action

$$GL_2(\mathbb{C}) \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$$

$$A \cdot [x_0 : x_1] = [ax_0 + bx_1 : cx_0 + dx_1]$$

induces a holomorphic action on \mathbb{H} by $SL_2(\mathbb{R})$ given by

$$A \cdot z = \frac{az + b}{cz + d}$$

If one defines an action on \mathbb{H} in this manner it is easy to verify that $A \cdot z \in \mathbb{H}$ – this follows from the fact that a, b, c , and d are all real – and that the action is holomorphic. Moreover, this is precisely how the matrices in $GL_2(\mathbb{C})$ act on \mathbb{H} (when they act holomorphically).

b. Show that the $SL_2(\mathbb{R})$ -action is transitive. This is quite trivial, actually. First, it can be shown that any purely imaginary number ri in the upper half plane can be sent to i via the map

$$\begin{pmatrix} 1/\sqrt{r} & 0 \\ 0 & \sqrt{r} \end{pmatrix}$$

Next, one need only observe that real translations are included in this group; they are represented by matrices of the form

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \cdot z = z + k$$

Therefore, given any number $z \in \mathbb{H}$ first translate z so that it is purely imaginary, then send that purely imaginary part to i :

$$z \mapsto z - \operatorname{Re}(z) = i\operatorname{Im}(z) = iy \mapsto \begin{pmatrix} 1/\sqrt{y} & 0 \\ 0 & \sqrt{y} \end{pmatrix} (iy) = i$$

Since everything in \mathbb{H} can be sent to i , this action is transitive.

c. Compute the stabilizer of i in $SL_2(\mathbb{R})$. Suppose A fixes i . Then

$$A \cdot i = i$$

$$\frac{ai + b}{ci + d} = i$$

$$\Rightarrow i(a - d) = b + c = 0$$

Since a, b, c , and d are real this implies that $a = d$ and $b = -c$. The determinant condition becomes $a^2 + b^2 = 1$, and so a stabilizer of i is of the form

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

where $a = \cos \theta$ and $b = \sin \theta$ for some $\theta \in [0, 2\pi]$.

d. Show that for the induced action of $SL_2(\mathbb{Z})$ on \mathbb{H} all stabilizers are finite cyclic groups.

Here we should make a note of a technical nature, namely that this action is **NOT** effective. This is true because the identity matrix I and its negative $-I$ represent the same automorphism of \mathbb{H} – In fact, this is true of every matrix A and $-A$ in this group. One would like to use theorems from the textbook, but, technically, these theorems require effective actions; this can be worked around in essentially two ways. One, instead of working in $SL_2(\mathbb{Z})$ we can work in $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{I = \pm I\}$. This group will now act holomorphically and effectively on \mathbb{H} , and we simply need to realize that certain quantities are reduced by a factor of $1/2$ – for example, an element of $PSL_2(\mathbb{Z})$ that has order 3 could have order 6 in $SL_2(\mathbb{Z})$.

Another direction to take is to determine which theorems can be applied to non-effective actions. For example, the proof of proposition 3.1 does not use the effective assumption, and so the result may be applied to non-effective actions. This theorem states that if G_p is finite, then G_p is cyclic. Therefore, to show that all stabilizers of $SL_2(\mathbb{Z})$ are finite cyclic, it suffices to show that they are all finite.

This result can be obtained by examining $SL_2(\mathbb{R})$ as a subset of \mathbb{R}^4 and noticing that the stabilizer for i will be a compact subset of \mathbb{R}^4 . This is true because the stabilizer of i is the continuous image of a compact set, namely, the unit circle in \mathbb{R}^2 :

$$\begin{aligned} S^1 &\rightarrow G_i(SL_2(\mathbb{R})) \\ \theta &\mapsto \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mapsto (\cos \theta, -\sin \theta, \sin \theta, \cos \theta) \end{aligned}$$

Therefore, the stabilizer of i is a compact subset of \mathbb{R}^4 . Now, let G_p be the stabilizer of $p \in \mathbb{H}$. As the action of $SL_2(\mathbb{R})$ is transitive, we have that $G_p = AG_iA^{-1}$ for some $A \in SL_2(\mathbb{R})$. This allows us to write G_p as the continuous (in fact smooth) image of a map from S^1 into \mathbb{R}^4 :

$$\theta \mapsto A \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} A^{-1} = (\text{linear combination of } \cos \theta \text{ and } \sin \theta, \dots)$$

Therefore, as subsets of \mathbb{R}^4 the stabilizers of $SL_2(\mathbb{R})$ are compact. Finally, as $SL_2(\mathbb{Z})$ is a discrete subset of $SL_2(\mathbb{R})$ it follows that a stabilizer in $SL_2(\mathbb{Z})$ will be a discrete subgroup

of a stabilizer in $SL_2(\mathbb{R})$. However, discrete subsets of compact sets are necessarily finite, which implies that any given stabilizer in $SL_2(\mathbb{Z})$ is finite.

e. Show that the only possible orders of these cyclic groups are 2, 4, and 6.

If we are working in $PSL_2(\mathbb{Z})$ this claim is not true; instead, all of the cyclic groups are of order 1, 2 or 3. However, let's work in $SL_2(\mathbb{Z})$. We first derive some necessary conditions on the order of a given stabilizer. Suppose $g \in SL_2(\mathbb{Z})$ stabilizes a given point p , and, further, let $g^{(k)} = 1$ where k is some positive integer.

We obviously have $g \in SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R})$ so that we may conjugate g by a matrix $A \in SL_2(\mathbb{R})$ where $A^{-1}gA \cdot i = i$. This means

$$A^{-1}gA = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$1 = A^{-1}g^{(k)}A = (A^{-1}gA)^k = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}^k = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$$

$$\begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, $\cos k\theta = 1$ and $\sin k\theta = 0$, or in other words $\theta \in (2\pi/k)\mathbb{Z}$. However, I claim that the parameter θ can only be in $\pi\mathbb{Z} \cup (\pi/2)\mathbb{Z} \cup (\pi/3)\mathbb{Z}$. This follows from the first equation above, which can be rewritten as

$$g = A \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} A^{-1}$$

Now, as $g \in SL_2(\mathbb{Z})$ it follows that we must be able to conjugate the stabilizer of i so that it has only integer entries. Multiplying everything out and using $ad - bc = 1$ to cancel some terms, we have

$$A \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} A^{-1} = \begin{pmatrix} \cos \theta + \sin \theta(ac + bd) & -\sin \theta(a^2 + b^2) \\ \sin \theta(d^2 + c^2) & \cos \theta - \sin \theta(ac + bd) \end{pmatrix}$$

Since every entry must be an integer, all linear combinations of the entries must be integers, and, in particular, we have that the $a_{11} + a_{22}$ entry must be an integer, which implies that $2 \cos \theta \in \mathbb{Z}$. This implies the result about possible values for θ .

We see that $\theta = 2\pi/k \in \pi\mathbb{Z} \cup (\pi/2)\mathbb{Z} \cup (\pi/3)\mathbb{Z}$ which implies that the possible values of k

are necessarily limited to 2, 4 or 6. However, does this mean there *are* elements of order 2, 4 and 6? The answer is yes: for an element of order 2 we have

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

which acts as the identity. For an element of order 4 we have

$$z \mapsto -\frac{1}{z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and for an element of order 6 we have

$$z \mapsto -\frac{1}{z+1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

Note: Because the action is not effective it is NOT true that the elements with non-trivial stabilizer form a discrete set. Notice that every element has non-trivial stabilizer as I and $-I$ fix all points. Also note that if we were to use $PSL_2(\mathbb{Z})$ the elements above would have orders 1, 2 and 3 respectively.