Math 428 Lecture Notes:
The functional equation for the Riemann ζ-function

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1 ζ defined

The Riemann ζ-function is defined

$$\zeta(s) = \sum_{n \in \mathbb{N}} n^{-s}.$$

We can bound

$$|\zeta(s)| \leq \sum_{n \in \mathbb{N}} n^{-\mathrm{Re}(s)} \leq 1 + \int_1^\infty x^{-\mathrm{Re}(s)} dx = 1 + \frac{x^{1-\mathrm{Re}(s)}}{1-\mathrm{Re}(s)}\Big|_1^\infty = \frac{\mathrm{Re}(s)}{\mathrm{Re}(s)-1} < \infty$$

provided $\mathrm{Re}(s) > 1$. Thus the infinite series converges absolutely in $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$ and uniformly on compact subsets, and $\zeta(s)$ is holomorphic in this region.

Our goal is to extend ζ to a meromorphic function on the whole complex plane. Such an extension is called a *meromorphic continuation*. We shall all describe a *functional equation* for ζ, which will govern its behavior in the left half-plane $\{s \in \mathbb{C} : \mathrm{Re}(s) < 0\}$.

2 Mellin transforms defined

The first tool we will require is the *Mellin transform*: If $f : \mathbb{R}^+ \to \mathbb{C}$ is a continuous, complex-valued function on the positive real line then we define

$$(Mf)(s) = \int_0^\infty f(x) x^s \frac{dx}{x}.$$

Theorem 2.1 *Suppose there exist real numbers $a < b$ and positive constants c_1, c_2 so that*

1. $|f(x)| < c_1 x^{-a}, x \to 0;$
2. $|f(x)| < c_2 x^{-b}, x \to \infty.$

Then the Mellin transform $(Mf)(s)$ is holomorphic in the region $U := \{s \in \mathbb{C} : a < \mathrm{Re}(s) < b\}$.

Proof: Fix $m \in \mathbb{N}$ so that the inequalities on f hold for $x < 1/m$ and $x > m$. Consider the sequence of functions

$$M_N f(s) = \int_{1/N}^{N} f(x) x^s \frac{dx}{x}$$

where $N > m$ is a positive integer. We have

$$\begin{aligned} |M_N f(s) - Mf(s)| &\leq \int_N^\infty |f(x)| x^{\mathrm{Re}(s)-1} dx + \int_0^{1/N} |f(x)| x^{\mathrm{Re}(s)-1} dx \\ &\leq c_2 \int_N^\infty x^{\mathrm{Re}(s)-1-b} dx + c_1 \int_0^{1/N} x^{\mathrm{Re}(s)-1-a} dx \\ &\leq c_2 \frac{x^{\mathrm{Re}(s)-b}}{\mathrm{Re}(s)-b}\Big|_N^\infty + c_1 \frac{x^{\mathrm{Re}(s)-a}}{\mathrm{Re}(s)-a}\Big|_0^{1/N} \\ &\leq \frac{c_2 N^{\mathrm{Re}(s)-b}}{b-\mathrm{Re}(s)} + \frac{c_1 N^{a-\mathrm{Re}(s)}}{\mathrm{Re}(s)-a} \to 0 \end{aligned}$$

provided $a < \mathrm{Re}(s) < b$, so $M_N f(s) \to Mf(s)$ uniformly on compact subsets of U.

It remains to show that each $M_N f(s)$ is holomorphic in U. Fix $s_0 \in U$, and γ a circular path of radius r centered at s_0, where r is chosen so that $\gamma \subset U$. Consider the power series expansion about s_0

$$x^{s-1} = \sum_{n \geq 0} a_n(x, s_0)(s - s_0)^n \quad a_n(x, s_0) = \frac{1}{2\pi i} \int_\gamma \frac{x^{w-1} dw}{(w - s_0)^{n+1}}$$

and observe that the coefficients are bounded

$$\begin{aligned}
|a_n(x,s_0)| &= \frac{1}{2\pi}\Big|\int_0^{2\pi} \frac{x^{re^{i\theta}+s_0-1} rie^{i\theta} d\theta}{(re^{i\theta})^{n+1}}\Big| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{x^{\mathrm{Re}(s_0)+r\cos\theta-1} d\theta}{r^n} \\
&\leq \begin{cases} x^{\mathrm{Re}(s_0)+r-1}/r^n \text{ if } x \geq 1 \\ x^{\mathrm{Re}(s_0)-r-1}/r^n \text{ if } x \leq 1 \end{cases}.
\end{aligned}$$

For s satisfying $|s-s_0|<r$, we have

$$M_N f(s) = \int_{1/N}^{N} f(x) \sum_{n\geq 0} (s-s_0)^n a_n(x,s_0) dx.$$

The integral converges absolutely in the sense that

$$\begin{aligned}
& \int_{1/N}^{N} \Big| f(x) \sum_{n\geq 0} (s-s_0)^n a_n(x,s_0) \Big| dx \\
\leq\ & \int_{1/N}^{N} |f(x)| \sum_{n\geq 0} |s-s_0|^n |a_n(x,s_0)| dx \\
\leq\ & \int_{1}^{N} |f(x)| x^{\mathrm{Re}(s_0)+r-1} \sum_{n\geq 0} (|s-s_0|/r)^n dx + \int_{1/N}^{1} |f(x)| \sum_{n\geq 0} (|s-s_0|/r)^n dx \\
\leq\ & \frac{r}{r-|s-s_0|}\Big[\int_{1}^{N} |f(x)| x^{\mathrm{Re}(s_0)+r-1} dx + \int_{1/N}^{1} |f(x)| x^{\mathrm{Re}(s_0)-r-1} dx\Big] \\
<\ & \infty.
\end{aligned}$$

It follows (e.g., by the Lebesgue dominated convergence theorem) that we can exchange summation and integration so that

$$M_N f(s) = \sum_{n\geq 0} (s-s_0)^n \int_{1/N}^{N} f(x) a_n(x,s_0) dx,$$

which means that $M_N f(s)$ is given by convergent power series near s_0, and thus is holomorphic. □

Remark 2.2 (A Generalization) With a bit more work, one can prove more:

Let $F(x,s)$ be a function of a real variable x and a complex variable s. Suppose there are open subsets $I \subset \mathbb{R}$ and $U \subset \mathbb{C}$ so that $F(x,s)$ is continuous on $I \times U$, $F(x_0, s)$ is holomorphic on U for each $x_0 \in I$, and

$$\int_I |F(x, s_0)| < \infty$$

for each $s_0 \in U$. Then the function

$$s \mapsto \int_I F(x,s)dx$$

is holomorphic on U.

Remark 2.3 A different (but correct!) proof was given in class. It used the following basic fact:

> [Lang, *Complex Analysis* V.1.2] Let γ be a path in an open set U and let g be a continuous function on γ. If s is not in γ, we define
>
> $$f(s) = \int_\gamma \frac{g(w)}{w-s} dw.$$
>
> Then f is holomorphic in $U \setminus \gamma$.

Proposition 2.4 (Inversion formula) *Retain the notation of Theorem 2.1, and let $c \in (a,b)$. Then we have*

$$f(x) = \int_{c-i\infty}^{c+i\infty} Mf(s) x^{-s} dx.$$

Proof:

$$\begin{aligned}
Mf(s) &= \int_0^\infty f(x) x^s \frac{dx}{x} \\
&= \int_{-\infty}^{\infty} f(e^y) e^{ys} dy \quad x = e^y \\
&= \int_{-\infty}^{\infty} f(e^y) e^{yc} e^{-2\pi i \xi y} dy \quad s = c - 2\pi i \xi \\
&= \hat{F}(\xi) = \hat{F}(\frac{i}{2\pi}(s-c))
\end{aligned}$$

where $\hat{F}$ is the Fourier transform of $F(y) = f(e^y)e^{cy}$.

Applying Fourier inversion:

$$\begin{aligned} F(y) &= \int_{-\infty}^{\infty} \hat{F}(\xi) e^{2\pi i \xi y} d\xi \\ &= \frac{i}{2\pi} \int_{c+i\infty}^{c-i\infty} \hat{F}(\frac{i}{2\pi}(s-c)) e^{2\pi i y \frac{i}{2\pi}(s-c)} ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Mf(s) e^{y(c-s)} ds \\ &= \frac{e^{cy}}{2\pi i} \int_{c-i\infty}^{c+i\infty} Mf(s) e^{-ys} ds. \end{aligned}$$

Dividing by e^{cy} and substituting $x = e^y$ gives the inversion formula.□

3 The Gamma function

The Γ function is defined

$$\Gamma(s) = \int_0^\infty x^s e^{-x} \frac{dx}{x}$$

and is holomorphic for $\mathrm{Re}(s) > 0$ by Theorem 2.1.

Proposition 3.1 (Basic Properties of Γ) *We have the formulas:*

1. $\Gamma(s+1) = s\Gamma(s)$;
2. $\Gamma(n+1) = n!$ *for* $n \in \mathbb{N}$;
3. $\Gamma(n+1/2) = (n-1/2)(n-3/2)\dots 1/2\sqrt{\pi}$.

Proof: First, note that

$$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 1$$

and

$$\begin{aligned} \Gamma(1/2) &= \int_0^\infty e^{-x} x^{-1/2} dx \\ &= \int_0^\infty e^{-y^2} 2dy \quad y^2 = x \\ &= \int_{-\infty}^\infty e^{-y^2} dy = \sqrt{\pi}. \end{aligned}$$

The last two statements then follow from the first statement. Applying integration by parts

$$\begin{aligned}\Gamma(s+1) &= \int_0^\infty x^{s+1}e^{-x}\frac{dx}{x}\\ &= (-x^s e^{-x})|_0^\infty - s\int_0^\infty x^s(-e^{-x})\frac{dx}{x}\\ &= 0 + s\Gamma(s). \quad \square\end{aligned}$$

Proposition 3.2 (Weierstrass Product) $\Gamma(s)^{-1}$ *extends to an function holomorphic on the whole complex plane. It has a Weierstrass product*

$$\Gamma(s)^{-1} = se^{\gamma s}\prod_{n=1}^\infty (1+s/n)e^{-s/n}, \quad \gamma := \lim_{n\to\infty}(\sum_{j=1}^n 1/j - \log n).$$

Proof: Recall the following formula from calculus

$$e^x = \lim_{n\to\infty}(1+x/n)^n, \quad x\in\mathbb{R}.$$

This suggests that we approximate Γ by the functions

$$\Gamma_n(s) := \int_0^n (1-x/n)^n x^s \frac{dx}{x},$$

which can be computed by standard calculus

$$\begin{aligned}\Gamma_n(s) &= \int_0^n (1-x/n)^n x^s\frac{dx}{x}\\ &= n^s\int_0^1 (1-y)^n y^{s-1}dy \quad x = ny\end{aligned}$$

integrating by parts

$$= n^s\Big[\frac{1}{s}\underbrace{(1-y)^n y^s|_0^1}_{=0} + \frac{n}{s}\int_0^1 (1-y)^{n-1}y^s dy\Big]$$

and iterating

$$= n^s\frac{n(n-1)\dots 2\cdot 1}{s(s+1)(s+2)\dots(s+n)}.$$

Inverting these approximating functions

$$g_n(s) := \Gamma_n(s)^{-1} = n^{-s}\frac{s(s+1)\dots(s+n)}{n(n-1)\dots 2\cdot 1}$$

yields a sequence of holomorphic functions.

We show that these converge uniformly on compact subsets, so the limit is a function holomorphic in the whole compact plane which coincides with $\Gamma(s)^{-1}$ for $\mathrm{Re}(s) > 0$. Consider the quotients

$$f_n(s) := g_n(s)/g_{n-1}(s) = (1-1/n)^s(1+s/n);$$

we interpret $g(s) := \lim_{n\to\infty} g_n(s)$ as an infinite product

$$g_1(s)\prod_{n=1}^{\infty} f_n(s).$$

We prove convergence for $s \in D_R$ for some $R \geq 1 \in \mathbb{R}^+$. Applying the inequality

$$|\log(1-z)+z)| < 2|z|^2 \text{ for } |z| \leq 1/2$$

we obtain

$$\begin{aligned} |\log f_n(s)| &= |s(\log(1-1/n)) + \log(1+s/n)| \\ &\leq 2|s|/n^2 + |-s/n + \log(1+s/n)| \\ &\leq 2|s|/n^2 + 2|s/n|^2 \leq 3(R/n)^2 \end{aligned}$$

for $n \gg 0$. Thus $\sum_{n\in\mathbb{N}} |\log f_n(s)| < \infty$ and the infinite product converges.

We regroup the terms in the product expression for g_n to make it look like a Weierstrass product

$$g_n(s) = se^{s(1+1/2+\ldots+1/n-\log(n))}\prod_{k=1}^{n}(1+s/k)e^{-s/k}.$$

In the limit as $n \to \infty$ we have

$$g(s) = se^{\gamma s}\prod_{k=1}^{\infty}(1+s/k)e^{-s/k}.$$

□

Proposition 3.3 (Functional equation) $\Gamma(s)$ *extends to a meromorphic function on the whole complex plane, with poles at the nonpositive integers, satisfying*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Proof: Let P_N be the Nth partial product for Γ^{-1}, so that $P_N(s)P_{N-1}(1-s)$ is equal to

$$\begin{aligned}
&= e^{\gamma} s \prod_{n=1}^{N}(1+s/n)e^{-s/n}(1-s)\prod_{n=1}^{N-1}(1+(1-s)/n)e^{(s-1)/n} \\
&= e^{\gamma} s \prod_{n=1}^{N}(1+s/n)e^{-s/n}(1-s)e^{-s/N}\prod_{n=1}^{N-1}(1-s/(n+1))e^{s/(n+1)}\frac{n+1}{n}e^{-1/n} \\
&= e^{\gamma+\log(N)-1-1/2-\ldots-1/(N-1)}e^{-s/N} s \prod_{n=1}^{N}(1+s/n)e^{-s/n}\prod_{m=1}^{N}(1-s/m)e^{s/m}.
\end{aligned}$$

Using the definition of γ and the Weierstrass product expansion

$$\sin \pi s = s\pi \prod_{n\neq 0}(1+s/n)e^{-s/n},$$

we conclude that $P_N(s)P_{N-1}(1-S)$ converges to $\frac{1}{\pi}\sin \pi s$ uniformly on compact subsets. □

4 Proof of the functional equation

Theorem 4.1 *The Riemann ζ function extends to a meromorphic function on the whole complex plane. The only pole is a simple pole at $s = 1$. If we set*

$$\xi(s) = \zeta(s)\Gamma(s/2)\pi^{-s/2}$$

then we have

$$\xi(s) = \xi(1-s).$$

Combining this with the functional equation for Γ, we obtain

Corollary 4.2

$$\zeta(s) = \pi^{s-3/2}\zeta(1-s)\Gamma(\frac{1-s}{2})\Gamma(1-\frac{s}{2})\sin(\pi s/2).$$

Proof of Theorem: The key ingredient in the proof is the functional equation for θ_3, specialized to $v = 0$:

$$\theta_3(0,-1/\tau) = \sqrt{\tau/i}\,\theta(0,\tau).$$

Consider the following auxiliary function

$$f(x) := \theta_3(0, ix) - 1 = 2\sum_{n\in\mathbb{N}} e^{-x\pi n^2},$$

which is real-valued for $x \in \mathbb{R}$.

Lemma 4.3 $|f(x)| \leq c_1 e^{-\pi x}$ for $x \to \infty$ and $|f(x)| \leq c_2 x^{-1/2}$ for $x \to 0$.

For the first part, let $y = e^{x\pi}$ so that $f = 2\sum_{n\in\mathbb{N}} y^{-n^2}$. For $y > 2$ we have

$$\sum_{n\in\mathbb{N}} y^{-n^2} \leq \sum_{n\in\mathbb{N}} y^{-n} = \frac{y^{-1}}{1-1/y} = \frac{1}{y-1} \leq 2/y$$

so $|f(x)| \leq 4e^{-\pi x}$ for $x \gg 0$. For the second part, we use the functional equation

$$f(x) = \theta_3(0, ix) - 1 = x^{-1/2}[1 + f(1/x)] - 1$$

so that $f(x) \sim x^{-1/2}$ as $x \to 0$.

We compute the Mellin transform $Mf(s/2)$, which by the Lemma and Theorem 2.1 is holomorphic for $1 < \mathrm{Re}(s) < \infty$:

$$\begin{aligned}
Mf(s/2) &= \int_0^\infty f(x)x^{s/2}\frac{dx}{x} \\
&= \underbrace{\int_1^\infty f(x)x^{s/2}\frac{dx}{x}}_{\text{holomorphic for all } s} + \int_0^1 f(x)x^{s/2}\frac{dx}{x}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 f(x)x^{s/2}\frac{dx}{x} &= \int_0^1 [x^{-1/2}(1 + f(1/x)) - 1]x^{s/2}\frac{dx}{x} \\
&\quad \text{by the functional equation} \\
&= \int_0^1 f(1/x)x^{(s-1)/2}\frac{dx}{x} + \int_0^1 x^{(s-1)/2}\frac{dx}{x} - \int_0^1 x^{s/2}\frac{dx}{x} \\
&\quad \text{setting } x = 1/y \\
&= \underbrace{\int_1^\infty f(y)y^{(1-s)/2}\frac{dy}{y}}_{\text{holomorphic for all } s} - \frac{2}{1-s} - \frac{2}{s}.
\end{aligned}$$

Thus we have

$$\xi(s) := \underbrace{\int_1^\infty f(x)\frac{1}{2}(x^{s/2} + x^{(1-s)/2})\frac{dx}{x}}_{\text{holomorphic for all } s} - \frac{1}{1-s} - \frac{1}{s}.$$

is holomorphic except for simple poles at 0 and 1, and is clearly symmetric under the transform $s \to 1-s$.

We compute the Mellin transform another way:

$$\begin{aligned} Mf(s/2) &= 2\int_0^\infty \sum_{n\in\mathbb{N}} e^{-x\pi n^2} x^{s/2}\frac{dx}{x} \\ &= 2\sum_{n\in\mathbb{N}}\int_0^\infty e^{-x\pi n^2} x^{s/2}\frac{dx}{x} \\ & \quad \text{substituting } y = x\pi n^2 \text{ in } n\text{-th term} \\ &= 2\sum_{n\in\mathbb{N}}(\pi n^2)^{-s/2}\int_0^\infty e^{-y}y^{s/2}\frac{dy}{y} \\ &= 2\pi^{-s/2}\zeta(s)\Gamma(s/2). \end{aligned}$$

Thus we have

$$\xi(s) = \pi^{-s/2}\zeta(s)\Gamma(s/2)$$

for $\mathrm{Re}(s) > 1$.

Of course, it follows that $\pi^{-s/2}\zeta s\Gamma(s/2)$ extends to a meromorphic function with just simple poles at 0 and 1. Using Proposition 3.2 and the formula

$$\zeta(s) = \pi^{s/2}\xi(s)\Gamma(s/2)^{-1},$$

we obtain an extension of ζ to a function meromorphic in the whole complex plane with just a simple pole at 1. □

5 Applications to values of ζ

Definition 5.1 *The* Bernoulli numbers *are the rational numbers defined by the Taylor expansion*

$$\frac{x}{e^x - 1} = 1 - x/2 + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{B_m}{(2m)!}x^{2m}.$$

For example

$$B_1 = \frac{1}{6} \quad B_2 = \frac{1}{30} \quad B_3 = \frac{1}{42} \quad B_4 = \frac{1}{30}.$$

Proposition 5.2 *For all $m \in \mathbb{N}$, we have*

$$\zeta(2m) = \frac{2^{2m-1}}{(2m)!} B_m \pi^{2m}.$$

For example

$$\zeta(2) = \frac{\pi^2}{6} \quad \zeta(4) = \frac{\pi^4}{90} \quad \zeta(6) = \frac{\pi^6}{945}.$$

Proof: Setting $x = 2iz$ in the Bernoulli Taylor series

$$\frac{2iz}{e^{2iz} - 1} = 1 - iz - \sum_{m=1}^{\infty} \frac{B_m 2^{2m}}{(2m)!} z^{2m}$$

which imples

$$z \cot z = 1 - \sum_{m=1}^{\infty} \frac{B_m 2^{2m}}{(2m)!} z^{2m}.$$

On the other hand,

$$\begin{aligned} z \cot z &= 1 + 2 \sum_{n=1}^{\infty} \frac{z^2}{z^2 - n^2 \pi^2} \\ &= 1 - 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{z^{2m}}{n^{2m} \pi^{2m}}. \end{aligned}$$

so comparing the corresponding terms gives the desired identity. □

Combining this with Corollary 4.2 and Proposition 3.1, we obtain

Proposition 5.3 *For all $m \in \mathbb{N}$, we have*

$$\zeta(1 - 2m) = (-1)^m \frac{B_m}{2m}.$$

For even negative integers, Corollary 4.2 gives:

Proposition 5.4 *For each $m \in \mathbb{N}$, $\zeta(-2m) = 0$.*

6 References

For the Γ-function and basic properties of Mellin transforms, see

Serge Lang, *Complex Analysis*, Springer-Verlag.

Our proof of the functional equation is derived from

David Mumford, *Tata Lectures on Theta I*, Birkhauser.

For more on values of ζ-functions at the integers see

Armand Borel, Values of zeta-functions at integers, cohomology and polylogarithms, *Current trends in mathematics and physics*, 1-44, Narosa, New Delha, 1995.