Math 428, Assignment 9: due November 25

1)Consider the congruence subgroup

$$\Xi = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : b \equiv c \equiv 0 \pmod 2 \quad or \quad a \equiv d \equiv 0 \pmod 2\}.$$

a)Show that Ξ is generated by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

b)Show that Ξ has exactly two cusps, ∞ and 1 (i.e., the set $\mathbb{Q} \cup \{\infty\}$ has two distinct orbits under the action of Ξ.)

2)a) Show that $\theta_3^8(0,\tau)$ is a modular form of weight 4 for the group Ξ, i.e.,

$$\theta_3^8(0, \frac{a\tau + b}{c\tau + d}) = (c\tau + d)^4 \theta_3^8(0,\tau) \text{ for each } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Xi$$

and $\theta_3^8(0,\tau)$ is holomorphic at the cusps ∞ and 1.

b)Consider the Eisenstein series

$$F_1(\tau) = \sum_{(m,n)\in\mathbb{Z}^2} ((m + 1/2) + n\tau)^{-4} \quad F_2(\tau) = \sum_{(m,n)\in\mathbb{Z}^2} (m + (n + 1/2)\tau)^{-4}.$$

Show that $F_1 + F_2$ is a modular form of weight 4 for the group Ξ.

c)Verify the identity

$$\theta_3^8(0,\tau) = \frac{3}{\pi^4}[F_1(\tau) + F_2(\tau)].$$

3)Let

$$r_8(n) = \#\{(n_1, \ldots, n_8) \in \mathbb{Z}^8 : n_1^2 + n_2^2 + \ldots + n_8^2 = n\}.$$

Prove that

$$r_8(n) = 16\Big[\sum_{d\in\mathbb{N}: n/d \text{ even integer}} d^3(-1)^d + \sum_{d\in\mathbb{N}: n/d \text{ odd integer}} d^3\Big].$$