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Brendan Hassett

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Proposition 0.1 *Given an angle $\angle CAB$ and a point D in its interior:*

1. *For each point E such that $E * A * C$, B is in the interior of $\angle DAE$.*
2. *B and C are on opposite sides of $\overleftrightarrow{AD}$.*

Proof: To say that D is in the interior of $\angle CAB$ means:

1. C and D are on the same side of $\overleftrightarrow{AB}$.
2. B and D are on the same side of $\overleftrightarrow{AC}$.

By the first Betweenness Axiom, $E * A * C$ means that these points are collinear, i.e.,

$$\overleftrightarrow{AE} = \overleftrightarrow{AC}$$

so we also have

B and D are on the same side of $\overleftrightarrow{AE}$.

To conclude that B is in the interior of $\angle DAE$, we just need to verify that

B and E are on the same side of $\overleftrightarrow{AD}$. $\dagger$

Since $C * A * E$, the segment CE meets $\overleftrightarrow{AD}$ in the point A, which means that

C and E are on opposite sides of $\overleftrightarrow{AD}$.

Consequently, an application of fourth Betweenness Axiom reduces the second assertion of the Proposition to †.

By definition, † holds provided the segment BE does not meet $\overleftrightarrow{AD}$. By the second Betweenness Axiom, there exists a point G so that $G * A * D$. By Proposition 3.4 (Line Separation Property), every point on $\overleftrightarrow{AD}$ lies on either $\overrightarrow{AD}$ or its opposite ray $\overrightarrow{AG}$. It suffices then to check that BE meets neither of these rays.

$BE \cap \overrightarrow{AD} = \emptyset$: All points of $\overrightarrow{AD}$ other than A lie on the same side of $\overleftrightarrow{AB}$. *Check this.* Similarly, all points of BE other than B lie on the same side of $\overleftrightarrow{AB}$. Now C and E are on opposite sides of $\overleftrightarrow{AB}$ so D and E are also on opposite sides of $\overleftrightarrow{AB}$, by the fourth Betweeness Axiom. It follows that points of BE (other than B) and points of $\overrightarrow{AD}$ (other than A) lie on opposite sides of $\overleftrightarrow{AB}$. In particular, BE and $\overrightarrow{AD}$ have no points in common.

$BE \cap \overrightarrow{AG} = \emptyset$: All points of $\overrightarrow{AG}$ (other than A) lie on the same side of $\overleftrightarrow{AC}$ and all points of BE (other than E) lie on the same side of $\overleftrightarrow{AC} = \overleftrightarrow{AE}$. By assumption, D and B lie on the same side of $\overleftrightarrow{AC}$, so G and B lie on opposite sides of $\overleftrightarrow{AC}$. Hence points of BE (other than E) and points of $\overrightarrow{AG}$ (other than A) lie on opposite sides of $\overleftrightarrow{AC}$, and BE and $\overrightarrow{AG}$ have no points in common. □