

Show that every group of order 15 is cyclic.

Let G be a group of order 15. By Cauchy's theorem, there exist elements x and y of order 3 and 5 respectively. It suffices to show that x and y commute with each other. Then we obtain a homomorphism

$$\begin{aligned}\phi : \mathbb{Z}_3 \times \mathbb{Z}_5 &\rightarrow G \\ (i, j) &\mapsto x^i y^j\end{aligned}$$

with trivial kernel. It follows that ϕ is an isomorphism and

$$G \simeq \mathbb{Z}_3 \times \mathbb{Z}_5 \simeq \mathbb{Z}_{15}.$$

The last isomorphism is an application of the general fact that $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$ whenever m and n are relatively prime.

Consider the action of G on itself by conjugation; fixed points of this action are just elements of the center of G . In our situation, it suffices to show that G has just one fixed point $\neq e$. Then G would have nontrivial center, which would force G to be abelian by exercise 37 of §15. Suppose there is no such fixed point: Each orbit of G under conjugation has order divisible by G , so the only possibilities are 3 and 5. Indeed, there must be one orbit with five elements and three orbits with three elements.

Let's analyze the orbit containing y . Set $z = xyx^{-1}$ and $w = x^{-1}yx$. If $z = y^j$ for some j then

$$\begin{aligned}y &= x^3 y x^{-3} \text{ because } x \text{ has order three} \\ &= x^2 y^j x^{-2} \\ &= x y^{j^2} x^{-1} \\ &= y^{j^3},\end{aligned}$$

which means $j^3 \equiv 1 \pmod{5}$ and $j = 1$. If $z = y$ then x and y commute and we are done.

If $z \neq y^j$ for any j , then z generates a new subgroup of order five, distinct from $\langle x \rangle$. Similarly, $w = x^{-1}yx$ also generates a subgroup of order five, distinct from $\langle x \rangle$. Furthermore, if we were to have

$$\langle z \rangle = \langle w \rangle$$

then $w = z^j$ for some j , whence

$$x^{-1}yx = xy^j x^{-1}$$

and multiplying x^2 and x^{-2} on the left and right would yield

$$xyx^{-1} = x^3y^jx^{-3} = y^j,$$

contradicting our assumption. It follows that

$$\{y, y^2, y^3, y^4, z, z^2, z^3, z^4, w, w^2, w^3, w^4\}$$

are all elements of order five, which contradicts our description of the sizes of the orbits of G under conjugation.